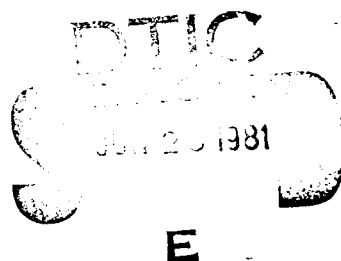


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ANNUAL REPORT  
ONR CONTRACT N00014-79-C-0073  
INVESTIGATION  
OF  
NONIDEAL PLASMA PROPERTIES



Prepared for: OFFICE OF NAVAL RESEARCH  
Covering Period: 1 November 1979 - 31 December 1980  
Principal Investigator: H. E. Wilhelm

1 May 1981

Department of Engineering Sciences  
University of Florida  
Gainesville, Florida 32611

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Acceleration Time	
Beam Current	X
Beam Energy	
Beam Size	
Beam Position	
Beam Angle	
Beam Quality	
Beam Stability	
Beam Lifetime	
Beam Losses	
Beam Monitoring	
Beam Control	
Beam Safety	
Beam Diagnostics	
Beam Research	
Beam Applications	
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## I. INTRODUCTION

This report contains theoretical investigations on the electrical conductivity and thermodynamic properties of nonideal plasmas, which were carried through in the period from 1 November 1979 to 31 December 1980 under ONR Contract N00014-79-C-0073. In addition, the theoretical results were compared with experimental data for nonideal plasmas. These comparisons are of a preliminary nature, since the experimental conductivities for nonideal plasmas differ not only quantitatively but also qualitatively in the literature.

CHAPTER II. The dependence of the electrical conductivity  $\sigma$  of dense (non-degenerate) plasmas on the nonideality parameters  $\gamma = Ze^2 n^{1/3} / KT$  was evaluated by summing the probabilities for  $v$ -body interactions ( $v = 2, 3, 4, \dots$ ) of the conduction electrons. It is shown that  $\sigma$  is noticeably smaller than the binary conductivity  $\sigma_2$  for  $\gamma > 10^{-1}$ . The theoretical decrease of  $\sigma$  with increasing  $\gamma$  is confirmed, however, only by some experimental data, while other experimental data indicate an increase of  $\sigma$  with increasing  $\gamma$  for the same pressure.

CHAPTER III. Based on the classical and quantum Boltzmann equations, the electrical conductivities of classical and degenerate nonideal plasmas were evaluated. Although in this kinetic approach many-body interactions are taken into account only through an exponentially shielded Coulomb potential, in which the electron-ion scattering occurs, the results give, in agreement with the experimental data, conductivities which are by about one order of magnitude smaller than the Spitzer conductivity for ideal plasmas. The increase of the dimensionless conductivity  $\sigma^* = m^{1/2} e^2 \sigma / (KT)^{3/2}$  with increasing  $\gamma$  is confirmed by some experimental data but not by all of them. The new Coulomb logarithm does no longer go to zero for large  $\gamma$  values (as in the Spitzer theory) but is well behaved for large electron densities, and even for solid state densities due to the consideration of electron degeneracy.

CHAPTER IV. With the help of quantum-field theoretical methods from the theory of metals, the electrical conductivity of nonideal plasmas was calculated under consideration of electron scattering by low-frequency plasmons (ion waves) and high-frequency plasmons (electron waves) for classical and degenerate conditions. The resulting conductivity formulas agree with the Spitzer theory for  $\gamma \rightarrow 0$  and exhibit numerical values which are considerably smaller than the Spitzer values but are still larger than the theoretical conductivities obtained in III for increasing  $\gamma$ . The numerical values  $\sigma^*$  agree with the experimental data qualitatively but are somewhat too high.

CHAPTER V. The possibility of anomalous diffusion and conduction transverse to magnetic fields  $\vec{B}_0$  was studied since large charged particle transport across magnetic fields is of interest for MHD generators. For weakly nonideal plasmas, the anomalous transverse conductivity was shown to be  $\sigma_{\perp} = \omega_p^2 / 4\pi\sqrt{2}\omega_B$ , where  $\omega_p = (ne^2/\epsilon_0 m)^{1/2}$  is the plasma frequency and  $\omega_B = |e|B_0/m$  is the gyration frequency of the electrons (e,m). This formula agrees with experimental data for weakly nonideal plasmas, but should be also correct qualitatively for nonideal plasmas. There are, however, no experimental data available on anomalous diffusion and conduction in magnetic fields for nonideal plasmas.

CHAPTER VI. In connection with the electric current transport in the electric field fluctuations produced collectively by the electrons and ions in random thermal motion, the electric microfield distribution of thermal plasmas was derived by equilibrium statistical mechanics. Comparison with the resulting temperature dependent microfield distribution with the classical (T-independent) Holtsmark distribution and its later extensions, indicates that the latter theories are approximately applicable to strongly nonideal plasmas but are invalid for ideal plasmas (to which they are usually applied in literature).

CHAPTERS VII - VIII. By means of Bose statistics, the contribution of the thermally excited (longitudinal) electron and ion waves to the free energy of nonideal classical and quantum plasmas was calculated. It is shown that the random low-frequency ion oscillations contribute more to the free energy than the high-frequency electron oscillations. The free energy of the random ion waves is quantitatively comparable to the free energy of the thermal (non- collective) ion motions for high densities ( $n < 10^{23} \text{ cm}^{-3}$ ) and standard plasma temperatures ( $T < 10^6 \text{ }^\circ\text{K}$ ). Similar calculations were performed for dense gases, in which the random sound oscillations lead, however, only to a small correction of the free energy.

The theoretical research on nonideal plasmas needs further clarifications by experiments. In particular more reliable conductivity data for nonideal alkali and noble gas plasmas are needed. This is a preliminary report of research results, which will be communicated later in form of publications.

## II. CONDUCTIVITY OF NONIDEAL PLASMAS WITH MANY-PARTICLE INTERACTIONS

By

H. E. Wilhelm

### ABSTRACT

The dependence of the electrical conductivity of nondegenerate, dense plasmas on the nonideality parameter,  $\gamma = Ze^2 n^{1/3} / kT$  (ratio of Coulomb interaction and thermal energies), is derived by summing the probabilities for  $\nu$ -body interactions ( $\nu = 2, 3, 4, \dots$ ) of the electrons. As an application, the dimensionless probability coefficients for binary and triple Coulomb interactions are calculated by means of simple physical models, and a conductivity formula for moderately nonideal plasmas ( $0 < \gamma < 1$ ) is derived in which all parameters are known. The theory is shown to agree with recent experimental data.

## INTRODUCTION

High pressure plasmas ( $10^1 \text{ bar} \lesssim P \lesssim 10^6 \text{ bar}$ ) produced by shock wave compression are now of considerable technical interest. A large number of publications<sup>1-10</sup> are concerned with the measurement of the anomalous electrical conductivity of proper nonideal plasmas ( $10^{-1} \lesssim \gamma \lesssim 1$ ). Theoretically, however, only the conductivity of ideal ( $\gamma \rightarrow 0$ ) and weakly nonideal ( $\gamma \ll 1$ ) plasmas is adequately understood.<sup>11,12</sup> The degree of nonideality of a fully ionized plasma is defined by the interaction parameter  $\gamma$ , which represents the ratio of average Coulomb interaction ( $Ze^2 n^{1/3}$ ) and thermal ( $KT$ ) energies ( $n$  = electron density,  $Z$  = ion charge number,  $e$  = elementary charge),

$$\gamma = Ze^2 n^{1/3} / KT = 1.670 \times 10^{-3} Z n^{1/3} T^{-1} \quad [\text{e.s.u.}] \quad (1)$$

The conductivity theories of ideal and weakly nonideal ( $0 < \gamma \ll 1$ ) plasmas break down for  $\gamma > 10^{-1}$ , since the Debye radius,

$$D = [Z/4\pi(1+Z)]^{1/2} \gamma^{-1/2} n^{-1/3} \sim \gamma^{-1/2} n^{-1/3}, \quad (2)$$

loses its physical meaning as an electric shielding and Coulomb interaction length. This is seen from the number of electrons  $N_D$  in the Debye sphere of a scattering ion, which is no longer large compared with one for  $\gamma > 10^{-1}$ ,

$$N_D = (4\pi/3)[Z/4\pi(1+Z)]^{3/2} \gamma^{-3/2} \sim \gamma^{-3/2}. \quad (3)$$

For strongly nonideal conditions,  $n > 10^{20} \text{ cm}^{-3}$  and  $T = 10^4 \text{ K}$ , we have  $\gamma > 0.775$ ,  $D < 4.881 \times 10^{-8} \text{ cm}$ , and  $N_D < 4.87 \times 10^{-2}$ . Another reason for the inapplicability of the conductivity theory of ideal and weakly nonideal plasmas to proper nonideal plasmas is the standard assumption of (shielded) binary Coulomb collisions ( $\nu = 2$ ), whereas, in reality, the conductivity is determined by many-particle interactions ( $\nu = 2, 3, 4, \dots$ ) for  $\gamma > 10^{-1}$ .

The many-body interaction is one of the classical, unsolved problems of physics. For this reason, we calculate the conductivity of nonideal plasmas and the probabilities for many-particle interactions by means of dimensional theory.<sup>13,14</sup> This approach gives the exact dependence on the relevant dimensional plasma parameters<sup>13,14</sup> and numerically correct results up to a dimensionless coefficient,<sup>13,14</sup> which is in general of the order  $10^0$ . The plasma is assumed to be fully ionized and nondegenerate, i.e.,

$$n = Zn_1 < \tilde{n} \quad , \quad \tilde{n} = 2(2\pi mKT/h^2)^{3/2} = 4.828 \times 10^{15} T^{3/2} \quad . \quad (4)$$

## ELECTRICAL CONDUCTIVITY

In a system of reference in which magnetic fields are absent, a linear electric current response  $\vec{j} = \sigma \vec{E}$  exists, provided that the generating electric field  $\vec{E}$  is weaker than the critical plasma field for electron heating. For any gaseous, liquid, or solid plasma, the electrical conductivity  $\sigma = |\vec{j}|/|\vec{E}|$  is given by

$$\sigma = (ne^2/m)\tau, \quad (5)$$

since the electrons of mass  $m \ll M$  dominate the electric current transport in plasmas. The interaction frequency  $\tau^{-1}$  of the electrons is the sum of the interaction frequencies  $\tau_v^{-1}$  for the  $v$ -particle interactions,

$$\tau^{-1} = \sum_{v=2}^N \tau_v^{-1}, \quad (6)$$

since the probabilities (frequencies)  $\tau_v^{-1}$  for many-particle interactions of the order  $v$  are additive ( $v = 2$  for binary,  $v = 3$  for ternary, etc.).  $N$  is related to the total number  $N^*$  of (charged) particles of the system by  $N = N^* - 1 \gg 1$ .

For physical reasons, the conductivity  $\sigma[\text{sec}^{-1}]$  of a fully ionized, classical plasma can depend only on the dimensional plasma parameters  $e[\text{cm}^{3/2}\text{gr}^{1/2}\text{sec}^{-1}]$ ,  $m[\text{gr}]$ ,  $n[\text{cm}^{-3}]$ ,  $KT[\text{gr cm}^2\text{sec}^{-2}]$ , and the characteristic dimensionless constant  $Z = n_1/n(KT = \text{thermal energy})$ . The conductivity  $\sigma$  and the parameters  $e$ ,  $m$ ,  $n$ , and  $KT$  have the dimensions  $\mathcal{D}(L = \text{dimension of length}, T = \text{dimension of time}, M = \text{dimension of mass})$ :

$$\mathcal{D}[\sigma] = T^{-1}, \quad \mathcal{D}[e] = L^{3/2}M^{1/2}T^{-1}, \quad \mathcal{D}[m] = M, \quad \mathcal{D}[n] = L^{-3}, \quad \mathcal{D}[KT] = ML^2T^{-2}. \quad (7)$$

Dimensional theory is based on the axioms of Dupré.<sup>15,16</sup> Accordingly, the secondary quantity  $\sigma$  is given in terms of the primary quantities  $e$ ,  $m$ ,  $n$ , and  $KT$  by<sup>15,16</sup>

$$\sigma = C_Z e^{N_1} m^{N_2} n^{N_3} (KT)^{N_4} \quad (8)$$

$C_Z$  is a dimensionless coefficient which depends on dimensionless parameters such as  $Z = n/n_1$ , and can only be determined by means of a detailed physical model.  $C_Z$  is either a true constant of order of magnitude one,  $C_Z \sim 1$ , or it is a slowly varying function (quasi-constant). Comparison of the powers of the independent dimensions  $L$ ,  $M$ , and  $T$  [Eq. (7)] in Eq. (8) gives the compatibility equations,

$$\frac{3}{2}N_1 - 3N_3 + 2N_4 = 0 \quad , \quad \frac{1}{2}N_1 + N_2 + N_4 = 0 \quad , \quad -N_1 - 2N_4 = -1 \quad (9)$$

These are three independent equations (since only three independent dimensions  $L$ ,  $M$ , and  $T$  exist) which determine three of the four powers  $N_i$  in terms of the fourth,

$$N_1 = 1 - 2N_4 \quad , \quad N_2 = -\frac{1}{2} \quad , \quad N_3 = \frac{1}{2} - \frac{1}{3}N_4 \quad , \quad N_4 \equiv N \quad (10)$$

Combining Eqs. (8) and (10) yields a conductivity expression  $\sigma = \sigma_N$ , which contains a still undetermined power  $N$ ,

$$\sigma_N = C_{ZN} (ne^2/m)^{1/2} (e^2 n^{1/3}/KT)^{-N} \quad (11)$$

In order to understand the physical meaning of Eq. (11), it is rewritten in the form of Eq. (5),

$$\sigma_v = ne^2/m\tau_v^{-1} \quad , \quad v = N/3 + 3/2 \quad , \quad (12)$$

where

$$\tau_v^{-1} = C_{Zv}^{-1} (ne^2/m)^{1/2} (e^2 n^{1/3}/KT)^{3(v - 3/2)} \quad , \quad v = 2, 3, 4, \dots, N \quad (13)$$

It is now seen that Eq. (11) or the equivalent Eqs. (12) and (13) represent the conductivity of a hypothetical plasma, in which each electron experiences only

many-body interactions of a fixed order  $\nu$ , since the probability for a  $\nu$ -body interaction has the  $n$ -dependence

$$\tau_{\nu}^{-1} \propto C_{Z\nu}^{-1} n^{\nu-1}, \quad \nu = 2, 3, 4, \dots, N. \quad (14)$$

For example, for a hypothetical plasma with 2-body interactions only (so-called ideal plasma), Eqs. (12) and (13) give

$$\sigma_2 = ne^2 / m \tau_2^{-1}, \quad \tau_2^{-1} = C_{Z2}^{-1} e^4 m^{-1/2} n (KT)^{-3/2}, \quad (15)$$

where the dimensionless coefficient is known from the kinetic theory of binary collisions,<sup>11</sup>  $C_{Z2} = (3/4)(2/\pi)^{1/2} / Z \ln \Lambda$ , i.e.,  $C_{Z2}$  is a quasi-constant which varies only slightly with  $n$  via the Coulomb logarithm  $\ln \Lambda$ .

Since the probabilities  $\tau_{\nu}^{-1}$  for the individual  $\nu$ -body interactions are additive, Eqs. (12) and (13) result in the following formulas for the interaction frequency  $\tau^{-1} = \sum \tau_{\nu}^{-1}$  and the conductivity  $\sigma = ne^2 / m \tau$  of actual plasmas, in which  $\nu = 2, 3, 4, \dots$ -body interactions take place:

$$\tau^{-1} = \sum_{\nu=2}^N \tilde{C}_{Z\nu}^{-1} \omega_p \gamma^{3(\nu-3/2)}, \quad (16)$$

$$\sigma = ne^2 / [m \sum_{\nu=2}^N \tilde{C}_{Z\nu}^{-1} \omega_p \gamma^{3(\nu-3/2)}], \quad (17)$$

where

$$\tilde{C}_{Z\nu} = (4\pi)^{1/2} Z^{3(\nu-3/2)} C_{Z\nu}, \quad \nu = 2, 3, 4, \dots, N, \quad (18)$$

$$\omega_p = (4\pi ne^2 / m)^{1/2}, \quad \gamma = Ze^2 n^{1/3} / KT \quad (19)$$

are dimensionless coefficients, the plasma frequency, and the nonideality parameter, respectively.

In order to expose the many-body effects ( $\nu \geq 3$ ) of the nonideal plasma for comparison with the corresponding formulas of binary kinetic theory ( $\nu = 2$ ),

Eqs. (16) and (17) are rewritten as

$$\tau^{-1} = m^{-1/2} e^4 n (KT)^{-3/2} [C_{Z2}^{-1} + \sum_{v=3}^N C_{Zv}^{-1} (\gamma/Z)^{3(v-2)}] , \quad (20)$$

$$\sigma = m^{-1/2} e^{-2} (KT)^{3/2} / [C_{Z2}^{-1} + \sum_{v=3}^N C_{Zv}^{-1} (\gamma/Z)^{3(v-2)}] . \quad (21)$$

Equations (20) and (21) give the interaction frequency and conductivity of nonideal plasmas in terms of a series in  $\gamma$ , which converges rapidly for  $0 < \gamma < 1$  and converges for any  $\gamma > 1$ , since it is finite ( $1 \ll N < \infty$ ),

$$\sum_{v=3}^N C_Z^{-1} (\gamma/Z)^{3(v-2)} = C_{Z3}^{-1} (\gamma/Z)^3 + C_{Z4}^{-1} (\gamma/Z)^6 + C_{Z5}^{-1} (\gamma/Z)^9 + \dots C_{ZN}^{-1} (\gamma/Z)^{3(N-2)} . \quad (22)$$

For ideal plasmas,  $\gamma \rightarrow 0$ , Eqs. (20) and (21) reduce to  $\tau^{-1} = \tau_2^{-1}$  and  $\sigma = \sigma_2$ , in accordance with the kinetic theory of binary interactions. For weakly nonideal plasmas,  $\gamma \ll 1$ , Eqs. (20) and (21) show that  $\tau^{-1} \gtrsim \tau_2^{-1}$  and  $\sigma \lesssim \sigma_2$ . For moderately nonideal,  $10^{-1} \lesssim \gamma \lesssim 10^0$ , and strongly nonideal,  $10^0 < \gamma < \tilde{\gamma}$ , plasmas, the electron interaction frequency  $\tau^{-1}$  increases, and the conductivity decreases considerably — and by orders of magnitude — respectively. These theoretical results are in agreement with measurements on nonideal plasmas,<sup>1-10</sup> which exhibit considerably smaller conductivities than expected from binary collision theory.

It should be noted that Eqs. (16) and (17) or (20) and (21) are applicable to nondegenerate plasmas only, i.e., to densities  $n < \tilde{n}$  or interaction parameters

$$0 < \gamma < \tilde{\gamma} , \quad \tilde{\gamma} = 2^{1/3} \frac{Ze^2/h}{(KT/2\pi m)^{1/2}} = 2.823 \times 10^2 ZT^{-1/2} . \quad (23)$$

For a physical interpretation of the above results, the partial collision frequency in Eq. (13) for the  $v$ -body interaction of a conduction electron with

$v-1$  other charged particles (e,i) is rewritten in the form

$$\tau_v^{-1} \sim (n\bar{Q}\bar{r})^{v-1} \bar{v}/\bar{r} \quad , \quad v = 2, 3, 4, \dots, N \quad , \quad (24)$$

where

$$\begin{aligned} n &\sim n_1 \quad , \quad Z \sim 1 \quad , \\ \bar{v} &\sim (KT/m)^{1/2} \quad , \\ \bar{r} &\sim e^2/KT \quad , \\ \bar{Q} &\sim (e^2/KT)^2 \end{aligned} \quad (25)$$

are, as to order of magnitude ( $\sim$ ), the electron or ion density, the speed of the conduction electron relative to the interaction partners (e,i), the Coulomb interaction radius, and the Coulomb scattering cross section, respectively ( $\bar{Q} = \pi\bar{r}^2$ ). The bar designates the thermal average.

In Eq. (24),  $\bar{Q}\bar{r}$  [ $\text{cm}^3$ ] is the interaction volume of one scattering partner,  $w = n\bar{Q}\bar{r}$  [1] is the spatial probability for a binary interaction,  $w^{v-1} = (n\bar{Q}\bar{r})^{v-1}$  [1] is the spatial probability for  $v-1$  simultaneous binary interactions (in a  $\Delta t \sim \bar{r}/\bar{v}$ ) of the conduction electron with  $v-1$  other charged particles ( $v$ -body interaction), and  $\bar{v}/\bar{r}$  [ $\text{sec}^{-1}$ ] is the frequency of interactions of the conduction electron, which occur at distances  $\bar{r}$ . Accordingly, the frequency for a  $v$ -body interaction of a conduction electron with  $v-1$  other charged particles is

$$\tau_v^{-1} = w^{v-1} \bar{v}/\bar{r} \quad , \quad w = n\bar{Q}\bar{r} \quad , \quad v = 2, 3, 4, \dots, N \quad . \quad (26)$$

## APPLICATION

For the practical use of the conductivity formula [Eq. (21)], the dimensionless coefficients  $C_{ZV}$  have to be determined either experimentally or by physical arguments, since a complete kinetic equation for many-particle Coulomb interactions is not available. For moderately nonideal plasmas, the conductivity is in good approximation given by Eq. (21) as

$$\sigma \cong \frac{(KT)^{3/2}}{m^{1/2} e^2 [C_{Z2}^{-1} + C_{Z3}^{-1} (\gamma/Z)^3]} \quad , \quad 0 < \gamma < 1 \quad , \quad (27)$$

since the next higher term in the denominator is of the order  $\gamma^6$ . Thus, for  $0 < \gamma < 1$ , it is sufficient to calculate the conductivity of nonideal plasmas from a physical model of two- and three-particle Coulomb interactions. In this case, only two dimensionless constants,  $C_{Z2}$  and  $C_{Z3}$ , have to be determined.

For ideal plasmas, the coefficient  $C_{Z2}$  has been evaluated by means of the Boltzmann equation for an unshielded Coulomb potential  $\phi = Ze/r$  (Rutherford scattering cross section) as  $C_{Z2} \propto Z^{-1}/\ln \Lambda$ , where  $\Lambda = [1 + b_{\max}/b_o]^2$ <sup>1/2</sup> and  $b_o = Ze^2/3KT$  (see, e.g., Ref. 11). Different authors prefer either the mean ion radius,<sup>11</sup>  $b_{\max} = (3Z/4\pi n)^{1/3}$ , or the Debye radius,<sup>12</sup>  $b_{\max} = D$ , as upper impact parameter in order to avoid the Coulomb divergence of the binary collision integral. Accordingly, either<sup>11</sup>  $\Lambda = [1 + 9(3Z/4\pi)^{2/3} \gamma^{-2}]^{1/2}$  or<sup>12</sup>  $\Lambda = [1 + 9(Z/4\pi(1+Z))\gamma^{-3}]^{1/2}$ . For both choices,  $\ln \Lambda \rightarrow 0$  for  $\gamma > 1$  and  $\ln \Lambda \ll 1$  for  $\gamma < 1$ .

For a classical nonideal plasma,  $0 < \gamma < \tilde{\gamma}$ , a physically meaningful Coulomb logarithm is obtained by evaluating the binary collision integral for a shielded Coulomb potential  $\phi = Ze \exp(-r/\delta)/r$  (Wentzel scattering cross section). This approach does not require an artificial cutoff of the impact parameter and gives<sup>15</sup>

$$C_{Z2} = (3/4)(2/\pi)^{1/2}/Z \ln \Lambda \quad , \quad (28)$$

where<sup>15</sup>

$$\Lambda = 8KT/(\hbar^2/m\delta^2) \gg 1 \quad (29)$$

for nondegenerate plasmas.  $\Lambda$  is proportional to the ratio of thermal ( $KT$ ) and quantum potential ( $\hbar^2/m\delta^2$ ) energies of the electrons, since the scattering in the shielded Coulomb potential is a wave-mechanical process (independent of  $n \leq \tilde{n}$ ).<sup>15</sup> Equation (29) contains not only the effects of binary interactions at distances  $r \lesssim \delta$  but also the collective many-body interactions at distances  $r > \delta$ .

The electric shielding length  $\delta$  of the Coulomb potential of the classical, nonideal plasma can depend only on  $Z$  and the dimensional parameters  $e$ ,  $m$ ,  $n$ , and  $KT$ . Dimensional analysis shows that a  $\delta_N$  with a still undetermined power  $N$  exists which is independent of  $m$ ,

$$\delta_N = C_{ZN} n^{-1/3} (e^2 n^{1/3}/KT)^{-N} \quad (30)$$

The interaction length is the linear superposition  $\delta = \delta_{N_1} + \delta_{N_2}$ , since  $\delta$  has to satisfy the limiting conditions,

$$\delta = [Z/4\pi(1+Z)]^{1/2} \gamma^{-1/2} n^{-1/3}, \quad \gamma \ll 1 \quad (N = 1/2), \quad (31)$$

$$\delta = (4\pi n/3Z)^{-1/3}, \quad \gamma \gg 1 \quad (N = 0), \quad (32)$$

corresponding to the Debye and mean ion radii, respectively. Elimination of the dimensionless constants by means of Eqs. (31) and (32) leads to the shielding length

$$\delta = (4\pi n/3Z)^{-1/3} [1 + (4\pi/3Z)^{1/3} (Z/4\pi(1+Z))^{1/2} \gamma^{-1/2}] \quad (33)$$

By Eq. (29),  $\Lambda \gg 1$  for  $0 < \gamma < 1$ , and  $\Lambda > \Lambda(\tilde{n}) \cong 4\pi(3Z/\pi)^{2/3}$  for  $1 < \gamma < \tilde{\gamma}$ .

It is seen that Eqs. (29) - (33) provide a satisfactory theory of the Coulomb logarithm of nondegenerate nonideal plasmas.

In order to calculate the coefficient  $C_{Z3}^{-1}$  for triple e-i-i interactions (e-i-e collisions are disregarded consistent with the disregard of e-e collisions in the evaluation of  $C_{Z2}^{-1}$ ), application is made of the model Eq. (24) which yields

$$\tau_v^{-1} \cong (n_i \bar{Q}_{ei} \bar{r}_{ei})^{\nu-1} \bar{v}_{ei} / \bar{r}_{ei}, \quad 0 < \gamma < 1, \quad (34)$$

where

$$\begin{aligned} n^i &= n/Z, \\ \bar{v}_{ei} &= (4/3)(8KT/\pi m)^{1/2}, \\ \bar{r}_{ei} &= Ze^2/3KT, \\ \bar{Q}_{ei} &= (\pi/4)(Ze^2/KT)^2 \ln \Lambda, \end{aligned} \quad (35)$$

are the exact thermal averages known from kinetic theory.<sup>15</sup> Comparison of Eq. (34) with Eq. (13) shows that

$$C_{Z\nu} \cong 4(8/\pi)^{1/2} \left(\frac{\pi}{12} Z^2 \ln \Lambda\right)^{\nu-1} Z^{-1}, \quad 0 < \gamma < 1. \quad (36)$$

Accordingly,  $C_{Z2}^{-1} \cong (4/3)(\pi/2)^{1/2} Z \ln \Lambda$  for  $\nu = 2$  in agreement with Eq. (28), and for  $\nu = 3$

$$C_{Z3}^{-1} = (\pi/9)(\pi/2)^{1/2} Z^3 (\ln \Lambda)^2, \quad 0 < \gamma < 1. \quad (37)$$

Substitution of Eqs. (28) and (37) into Eq. (21) yields for the electrical conductivity of moderately nonideal plasmas:

$$\sigma \cong \frac{(3/4)(2/\pi)^{1/2} (KT)^{3/2}}{m^{1/2} Ze^2 [\ln \Lambda + (\pi/12) Z^{-1} (\ln \Lambda)^2 \gamma^3]}, \quad 0 < \gamma < 1. \quad (38)$$

The Coulomb logarithm,  $\ln \Lambda$ , is evaluated in Eq. (29) in dependence of  $n$ ,  $T$ , and  $\gamma$ .

Equations (34), (36), and (37) are based on binary e-i and triple e-i-i collisions and collective many-particle interactions which are considered through the shielded Coulomb potential with the interaction length  $\delta = \delta(n, \gamma)$ , Eq. (33). The approximately equal signs in these equations are a reference to the disregard of binary (e-e) and triple (e-i-e) collisions. Furthermore, we have restricted the applicability of the results to moderately nonideal ( $0 < \gamma < 1$ ) plasmas, since the Coulomb logarithm has been calculated without considering the influence of triple collisions. The latter effect is in all probability not quantitatively significant since  $\ln \Lambda$  is a slowly varying function of  $\Lambda$ . A more accurate determination of the dimensionless coefficients  $C_{22}$  and  $C_{23}$  has to be postponed until a kinetic equation for nonideal plasmas is available, which takes into account not only binary but at least also triple interactions and correlations.

The experimental data for nonideal alkali and noble gas plasmas<sup>10</sup> ( $0.1 < \gamma < 1$ ) indicate that the electrical conductivity is roughly an order of magnitude smaller than predicted by the theories of ideal<sup>11</sup> and weakly nonideal<sup>12</sup> plasmas. In Fig. 1, isobars of the dimensionless Coulomb conductivity  $\sigma^* = m^{1/2} e^2 \sigma / (KT)^{3/2}$  are reproduced versus the number of electrons in the Debye sphere,  $N_D = 4\pi D^3 n / 3 \sim \gamma^{-3/2}$  [Eq. (3)], showing (1) conductivities according to the ideal plasma theory<sup>11</sup> ( $b_{\max} = D$ ), (2) computer conductivities from molecular dynamics and Monte Carlo methods (with error estimates) by Valuev and Norman,<sup>16</sup> and (3) experimental conductivities for a cesium plasma at a pressure  $p = 5 \times 10^4$  Pa by Dikhter *et al.*<sup>10</sup> Conductivity curves based on the present analytical theory [Eq. (38)] are shown for (4) lithium and (5) cesium plasmas at a pressure  $p = 5 \times 10^4$  Pa for comparison.

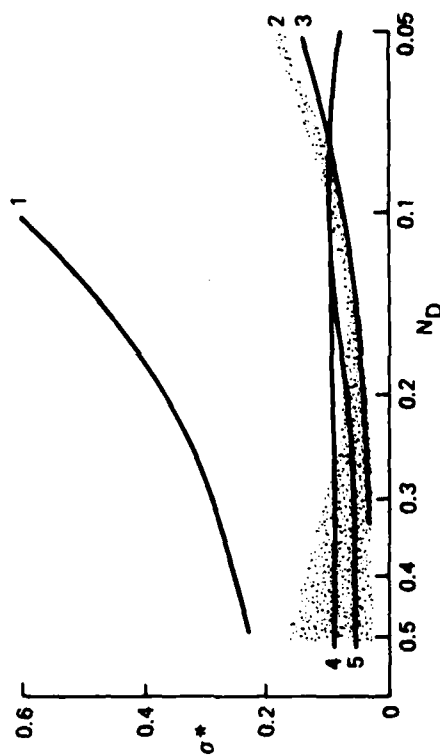


FIG. 1: Dimensionless conductivity  $\sigma^*$  versus number  $N_D$  of particles in the Debye sphere: (1) ideal plasma theory<sup>11</sup>, (2) molecular dynamics machine calculations<sup>16</sup>, (3) experimental data for Cs plasmas<sup>10</sup> at  $p = 5 \times 10^4$  Pa, (4) and (5) present theory for Li and Cs plasmas ( $p = 5 \times 10^4$  Pa).

Figure 1 demonstrates that the theoretical conductivity values from Eq. (38) are correct as to order of magnitude and lie well within the errors of the experimental data<sup>10,17</sup> and the computer experiments.<sup>16</sup> Equation (38) predicts a slight decrease of  $\sigma^*$  for  $N_D < 0.1$ , which is due to the contributions from the triple interactions [Fig. 1, curves (4) and (5)]. This effect was observed by Kulik et al.<sup>17</sup> in measurements on cesium plasmas.

## REFERENCES

- \* Supported by the U. S. Office of Naval Research.
- 1. E. U. Franck and F. Hensel, Phys. Rev. 147, 109 (1966).
- 2. V. A. Alekseev, Teplofiz. Vys. Temp. 8, 689 (1970).
- 3. N. V. Ermokhin, B. M. Kovalev, P. P. Kulik, and V. A. Ryabiy, Teplofiz. Vys. Temp. 9, 665 (1971).
- 4. B. N. Lomakin and V. E. Fortov, Sov. Phys. JETP 36, 48 (1973).
- 5. S. H. Barolskii, N. V. Ermokhin, P. P. Kulik, and V. M. Melnikov, Sov. Phys. JETP 35, 94 (1976).
- 6. Yu. V. Ivanov, V. B. Mintsev, V. E. Fortov, and A. N. Dremine, Sov. Phys. JETP 44, 112 (1976).
- 7. C. Goldbach, G. Nollez, S. Popovic, and M. Popovic, Z. Naturforsch. 33a, 11 (1977).
- 8. N. V. Ermokhin, B. M. Kovalev, P. P. Kulik, and V. A. Ryabiy, Teplofiz. Vys. Temp. 15, 695 (1977).
- 9. G. E. Norman and L. Yn. Schnrova, Teplofiz. Vys. Temp. 17, 863 (1979).
- 10. I. Ya. Dikhter, V. A. Zeigarnik, and S. V. Smagin, Teplofiz. Vys. Temp. 17, 256 (1979).
- 11. S. Chapman and T. G. Cowling, The Mathematical Theory of Nonuniform Gases (University Press, Cambridge, 1939).
- 12. W. Ebeling and G. Roepke, Ann. Phys. 36, 429 (1979).
- 13. J. Dupré, Théorie Mécanique de la Chaleur (Masson & Cie, Paris, 1869).
- 14. E. Buckingham, Phys. Rev. 4, 345 (1914).
- 15. H. E. Wilhelm, Phys. Rev. 187, 382 (1969).
- 16. A. A. Valuev and G. E. Norman, Teplofiz. Vys. Temp. 15, 689 (1975).
- 17. P. P. Kulik, E. K. Rozanov, and V. A. Ryabiy, Teplofiz. Vys. Temp. 15, 415 (1977).

### III. CONDUCTIVITY OF NONIDEAL CLASSICAL AND QUANTUM PLASMAS

By

H. E. Wilhelm

#### ABSTRACT

The electrical conductivity of fully ionized, moderately nonideal plasmas with Coulomb interaction parameters  $0.1 < \gamma \leq 1$ , where  $\gamma = Ze^2 n^{1/3} / kT$  is the ratio of Coulomb and thermal energies, is calculated for displaced Maxwell and Fermi electron distributions, respectively. The electrons are scattered by an effective Coulomb potential  $\phi(r) = Ze r^{-1} \exp(-r/\delta)$ , which considers binary ( $0 < r < \delta$ ) and many-body ( $\delta < r < \infty$ ) interactions. The shielding distance is given by  $\delta = \alpha (4\pi n / 3Z)^{-1/3}$  with  $\alpha = \alpha_0 \gamma^{-N-1}$  for classical plasmas and  $\delta = \beta (4\pi n / 3Z)^{-1/3}$  with  $\beta = \beta_0 \gamma^{-N} \Gamma^{-M-1}$  for quantum plasmas, where  $\Gamma = Ze^2 n^{1/3} / \hbar^2 m^{-1} n^{2/3}$  is the ratio of Coulomb interaction and quantum potential energies of the electrons. It is shown that the resulting conductivity formulas are applicable to densities up to four orders of magnitude higher than those of the ideal conductivity theory, which breaks down at higher densities because the Debye radius loses its physical meaning as a shielding length and upper impact parameter.

## INTRODUCTION

The theory of the electrical conductivity of fully ionized plasmas<sup>1-3)</sup> based on the Boltzmann equation, the Fokker-Planck equation (derived by expanding the binary collision integral for the small, successive velocity changes of Coulomb scattering), or the Lenard-Balescu equation (taking into account the dielectric properties of the medium) is in agreement with the experimental data for rarefied high-temperature plasmas,  $\gamma \ll 1$ . The interaction parameter is defined as the ratio of (average) Coulomb interaction ( $Ze^2 n^{1/3}$ ) and thermal (KT) energies (n is the electron density and Z the ion charge number),

$$\gamma = Ze^2 n^{1/3} / KT = 1.670 \times 10^{-3} Z n^{1/3} / T$$

in cgs-units which will be used throughout. The conventional transport calculations<sup>1-3)</sup> give an electrical conductivity of the form  $\sigma \sim (KT)^{3/2} / m^{1/2} e^2 Z \ln \Lambda_D$  for classical ideal plasmas, where  $\Lambda_D = [1 + (D/p_0)^2]^{1/2} \approx D/p_0$  for  $D \gg p_0$ . D is the maximum impact parameter for which the Debye length is used, and  $p_0$  is the average impact parameter for 90° deflections (Landau length),  $p_0 = Ze^2 / 2KT$ . The condition,  $\Lambda_D \gg 1$  or  $\ln \Lambda_D \sim 10^1$  is satisfied only for not too low temperatures T and not too high densities n.<sup>4)</sup> Conductivity formulas with this Coulomb logarithm break down for large interaction parameters  $\gamma$  and densities n, since the Debye radius

$$D = [Z/4\pi(1 + Z)]^{1/2} \gamma^{-1/2} n^{-1/3}$$

becomes smaller than the atomic dimension  $10^{-8}$  cm and, thus, completely loses its physical meaning as an electric shielding length and maximum impact parameter. E.g., for  $T = 10^4$  °K,  $\gamma > 10^0$  and  $D < 10^{-8}$  cm if  $n > 10^{20}$  cm<sup>-3</sup>. Moderately nonideal plasmas with  $\gamma \sim 1$  are readily generated through shock wave compression and exhibit conductivities of the order  $\sigma \sim 10^1 - 10^2$  mho/cm<sup>5-6)</sup>, which are much smaller than those which would be obtained by applying the conductivity formula for ideal plasmas in the nonideal regime.

Although there are some bulk measurements of the electrical conductivity of nonideal cesium and noble gas plasmas available<sup>5-8)</sup>, theoretical explanations of these results are still missing. The momentum and energy transport in weakly nonideal plasmas,  $\gamma \ll 1$ , was treated by Wilhelm<sup>9)</sup> by means of an exponentially shielded Coulomb potential, which permits to consider not only short-range binary ( $r \leq D$ ) but also long-range many-body ( $r > D$ ) interactions. This interaction model was used shortly afterwards by Rogov<sup>10)</sup> for the calculation of the conductivity of weakly, nonideal argon and xenon plasmas with Debye shielding.

For moderately nonideal plasmas,  $0.1 < \gamma \leq 1$ , various phenomenological approaches have been used to extend the conductivity formula of ideal plasmas, e.g., Goldbach et al<sup>11)</sup> multiply the Debye length  $D$  with a free parameter  $x(p)$  which is chosen to match the experimental data, i.e. to compensate for the too rapid decrease of  $D$  with pressure. A kinetic equation has been proposed for nonideal plasmas by Klimontovich<sup>12)</sup>, which considers spatial correlations and temporal retardation in the collision integrals. This equation appears to have not yet lead to transport coefficients because of the mathematical difficulties associated with its solution.

In the following, the momentum relaxation time and the electrical conductivity of (i) classical and (ii) quantum plasmas is calculated for intermediate nonideal conditions,  $0.1 < \gamma \leq 1$ . For this region of interaction, the concept of Debye shielding already breaks down since the number of particles in the Debye sphere  $4\pi D^3/3$  is no longer large compared with one for  $\gamma > 0.1$ . This difficulty can not be remedied by replacing  $D$  with the quantum mechanical shielding length<sup>13)</sup> ( $\hbar = 2\pi\hbar = \text{Planck constant}$ ),

$$D_F = (\pi a_0 / 4k_F)^{1/2}, \quad a_0 = \frac{\hbar^2}{me^2}, \quad k_F = 2\pi \left( \frac{3n}{8\pi} \right)^{1/3}$$

which is too small as  $D$  in most high pressure plasmas, e.g.  $D_F \sim 10^{-8}$  cm for  $n = 10^{20}$  cm<sup>-3</sup>. From the definition of the mean particle distance, it is clear that the mean ion distance  $\delta \sim n_i^{-1/3}$  separates the region in which an electron

experiences few-body encounters ( $r \leq \delta$ ) from the region in which an electron experiences many-body interactions ( $r > \delta$ ) in a nonideal plasma, as long as  $\delta > 10^{-8} \text{ cm}$  ( $n_1 < 10^{24} \text{ cm}^{-3}$ ). Thus, the mean ion distance evolves naturally as the characteristic interaction distance for nonideal plasmas, for which Debye and Fermi shielding fail. We demonstrate mathematically that  $\delta \approx \gamma^{-N} n_1^{-1/3}$  with  $0 \leq N < \frac{1}{2}$  for classical plasmas and  $\delta \approx \gamma^{-N} \Gamma^{-M} n_1^{-1/3}$  with  $0 \leq M \leq \frac{1}{2}$  for quantum plasmas ( $\Gamma = Ze^2 n^{1/3} / \hbar^2 m^{-1} n^{2/3}$ ), i.e.  $\gamma^{-N} \sim 1$  and  $\Gamma^{-M} \sim 1$  are correction factors which are insignificant since the plasma conductivity depends logarithmically on  $\delta$ .

We calculate the electrical conductivity of plasmas with (i) Maxwell and (ii) Fermi distributions of the electrons, when all ions have the same charge number  $Z$ . The electrons are assumed to be scattered by the exponentially shielded Coulomb potential  $\phi = Ze r^{-1} \exp(-r/\delta)$  which takes many-body interactions at distances  $r > \delta$  into account. The considerations are applicable only to moderately nonideal conditions,  $0.1 < \gamma \leq 1$ , up to densities  $n \ll 10^{24} \text{ cm}^{-3}$ . Thus, the following theory is limited to densities  $n$  well below the electron density in (solid) metals.

## PHYSICAL FOUNDATIONS

The electrical conductivity  $\sigma$  of any gaseous, liquid, or solid medium, in which the electrical current transport is due to electrons, is proportional to the electron density  $n$  and the relaxation time  $\tau$  of the average momentum  $\langle m\vec{v}_e \rangle$  of the electrons ( $m$  is the electron mass and  $e > 0$  is the elementary charge)

$$\sigma = (ne^2/m)\tau. \quad (1)$$

The relaxation time  $\tau$  is determined by the scattering potential and the (classical or quantum statistical) kinetics of the electron gas in the electric field.

In nonideal plasmas, the region  $0 < r \leq \delta$  of binary and few - body collisions and the region  $\delta < r < \infty$  of many - body interactions are separated by the electric shielding radius  $\delta$ . Dimensional theory gives for classical ( $n \ll \tilde{n}$ ) and quantum ( $n \gtrsim \tilde{n}$ ) plasmas (see Appendix) :

$$\delta = \alpha(3Z/4\pi n)^{1/3}, \quad n \ll \tilde{n}, \quad (2)$$

$$\alpha = \alpha_0 \gamma^{-N}, \quad 0 \leq N \leq \frac{1}{2}, \quad (3)$$

and

$$\delta = \beta(3Z/4\pi n)^{1/3}, \quad n \lesssim \tilde{n}, \quad (4)$$

$$\beta = \beta_0 \Gamma^{-M} \gamma^{-N}, \quad 0 \leq M, N \leq \frac{1}{2}, \quad (5)$$

where

$$\alpha_0 = Z^N (4\pi/3Z)^{1/3} / [4\pi(1+Z)]^{1/2}, \quad \beta_0 = Z^M \alpha_0, \quad (6)$$

$$\tilde{n} = 2(2\pi m kT/h^2)^{3/2}. \quad (7)$$

The nonideality parameters of the classical and completely degenerate plasmas are defined by

$$\gamma = Ze^2 n^{1/3} / kT, \quad \Gamma = Ze^2 n^{1/3} / \hbar^2 m^{-1} n^{2/3}. \quad (8)$$

The classical formulas (2)-(3) are the special case  $M = 0$  ( $\beta_0 = \alpha_0$ ) of the general quantum-mechanical Eqs. (4)-(5). Eq.(2) becomes the Debye radius<sup>4)</sup> of the weakly nonideal ( $\gamma \ll 1$ ) classical plasma for  $N = \frac{1}{2}$ ,  $\delta(N=\frac{1}{2})=D$ , whereas Eq.(4) becomes the Fermi radius<sup>13)</sup> of the completely degenerate ( $n \gg \tilde{n}$ ) plasma for  $M = \frac{1}{2}$  and  $N = 0$ ,  $\delta(M=\frac{1}{2}, N=0) \sim D_F$ . For  $M=N=0$ , Eqs. (2) and (4) reduce to the shielding radius of the strongly nonideal plasma, in which the kinetic energy ( $KT$  or  $\hbar^2 m^{-1} n^{2/3}$ ) of the electrons is negligible,  $\delta(M=N=0) \sim (3Z/4\pi n)^{1/3}$ , which is the mean ion distance up to a factor  $\alpha_0 = \beta_0 \sim 1$ . For these reasons, the powers  $M$  and  $N$  in Eqs.(2)-(6) are limited to the interval  $0 \leq M, N \leq \frac{1}{2}$ . For non-solid ( $n \ll 10^{24} \text{ cm}^{-3}$ ) plasmas of intermediate nonideality,  $0.1 < \gamma \leq 1$ , which implies  $1 < \Gamma \leq 10^2$  since  $\Gamma = (KT/\hbar^2 m^{-1} n^{2/3})\gamma$ , extremely simple relations hold as to order-of-magnitude:

$$\alpha(\gamma) \sim 1, \quad \beta(\gamma, \Gamma) \sim 1, \quad \delta \sim (3Z/4\pi n)^{1/3}. \quad (9)$$

Based on the above considerations, the scattering of electrons by  $Z$ -times charged ions in plasmas of intermediate nonideality is described by the shielded Coulomb potential

$$\phi(r) = Ze r^{-1} \exp(-r/\delta), \quad 0 < r < \infty, \quad 0.1 < \gamma \leq 1. \quad (10)$$

which contains the binary and few - body collisions at distances  $0 < r \leq \delta$  and the many - body interactions at distances  $\delta < r < \infty$ . A similar Coulomb potential is used in the conductivity theory of metals, although the use of such a "binary quasi-potential" is questionable for densities  $n > 10^{22} \text{ cm}^{-3}$ .

The differential cross section  $\sigma(\theta, g)$  for the scattering ( $\vec{g} \rightarrow \vec{g}^*$ ) of electrons by the potential (10) is in the center of mass system<sup>14)</sup>

$$\sigma(\theta, g) = (Ze^2/2m)^2 / [g^2 \sin^2(\theta/2) + u^2]^2, \quad u = \hbar/2m\delta \quad (11)$$

where  $\theta = \angle(\vec{g}, \vec{g}^*)$ ,  $\vec{g} = \vec{v}_e - \vec{v}_i$ ,  $\vec{g}^* = \vec{v}_e^* - \vec{v}_i^*$ , and

the electron and ion velocities before and after the interaction are designated by  $\vec{v}_{e,i}$  and  $\vec{v}_{e,i}^*$ , respectively. The speed  $u$  corresponds to a de Broglie wave length of the order  $\lambda \sim \delta$ . For  $u \rightarrow 0$  or  $\delta \rightarrow \infty$ , Eq. (11) reduces to the Rutherford cross section.<sup>14)</sup>

The scattering cross section  $\sigma(\theta, g)$  is strictly valid only in the Born approximation<sup>14)</sup>. Contrary to what one might expect in general for the latter, Eq. (11) describes in good approximation the scattering in the exponentially decaying potential (10) because of the peculiarity of the Coulomb interaction. The Coulomb interaction  $\phi \sim 1/r$  has the unique property that the Born approximation and the exact wave mechanical approach give the same scattering cross section<sup>14)</sup> (identical with the Rutherford formula). In the region  $0 < r < \delta$ , the interaction potential (10) is practically Coulombic, and thus the Born approximation gives the correct solution. In the region  $\delta < r < \infty$ , the interaction potential (10) is effectively screened, i.e., the Born approximation gives the correct solution because  $\phi(r)$  is small. In the transition zone  $r \approx \delta$ , the Born approximation holds fairly well for reasons of continuity.

The relaxation time  $\tau$  is obtained by evaluation of the collision integrals for the electron momentum  $m\vec{v}_e$  for the (i) classical and (ii) degenerate plasma, respectively. Both in the cases of classical and Fermi statistics, the particle velocities  $\vec{v}_{e,i}$  and  $\vec{v}_{e,i}^*$  before and after the interaction are interrelated by the classical conservation equations for momentum and energy.

# CONDUCTIVITY OF CLASSICAL PLASMA

According to kinetic theory, the average momentum density  $nm(\langle \vec{v}_e \rangle - \langle \vec{v}_i \rangle)$  exchanged per unit time between electrons and ions, interacting with the Coulomb potential (10), is given by the collision integral for  $m\vec{v}_e$ , which determines the momentum relaxation time  $\tau$ ,

$$-nm(\langle \vec{v}_e \rangle - \langle \vec{v}_i \rangle)/\tau = m \int \dots \int \vec{v}_e [f_e(\vec{v}_e^*) f_i(\vec{v}_i^*) - f_e(\vec{v}_e) f_i(\vec{v}_i)] g \sigma(\theta, g) d\Omega d\vec{v}_e d\vec{v}_i. \quad (12)$$

The scattering cross section  $\sigma(\theta, g)$  is given in Eq.(11) and the solid angle element is  $d\Omega = \sin \theta d\theta d\phi$ . In response to an applied electric field  $\vec{E}$ , the electrons and ions drift with velocities  $\langle \vec{v}_e \rangle$  and  $\langle \vec{v}_i \rangle$  so that their distribution functions are displaced Maxwellians,

$$f_s(\vec{v}_s) = n_s (m_s/2\pi K T_s)^{3/2} \exp[-\frac{1}{2} m_s (\vec{v}_s - \langle \vec{v}_s \rangle)^2 / K T_s], \quad s=e, i. \quad (13)$$

Eq.(13) represents a 5-moment-approximation to the nonequilibrium solution of the Boltzmann equation. The perturbations of  $f_s(\vec{v}_s)$  due to viscous stresses and heat flows are neglected in Eq.(13), since they yield only corrections of higher order to the conductivity.

The collision integral (12) is integrated by standard methods<sup>9)</sup> for subsonic drift velocities,  $|\langle \vec{v}_e \rangle - \langle \vec{v}_i \rangle| < (2KT/m)^{1/2}$ , with the usual approximations  $(m_{ei} = m_e m_i / (m_e + m_i) \approx m_e \equiv m, T_{es} = m_{es} [(T_e/m_e) + (T_i/m_i)] \approx T_e \equiv T)$ . (For supersonic drift velocities, a linear response  $\vec{j} = \sigma \vec{E}$  between current density  $\vec{j}$  and electric field  $\vec{E}$  does no longer exist.<sup>9)</sup>) The resulting relaxation time is given by:<sup>17)</sup>

$$\tau^{-1} = \frac{8}{3} (2KT/\pi m)^{1/2} n_i Q, \quad (14)$$

$$Q = \frac{\pi}{4} (Ze^2 / KT)^2 L, \quad (15)$$

$$L = e \Lambda^{-1} E_1 (\Lambda^{-1}), \quad (16)$$

where

$$\Lambda = 2KT/\mu u^2 = \alpha^2 (8m/\hbar^2) (4\pi n/3Z)^{-2/3} / KT \quad (17)$$

by Eqs.(2) and (11) for the classical plasma. Furthermore

$$E_1(x) = -\Gamma^* - \ln x - \sum_{m=1}^{\infty} (-1)^m x^m / m(m!) \quad (18)$$

is the exponential integral of order one ( $\Gamma^* = 0.477.. = \text{Euler's constant}$ ).<sup>18)</sup>

The latter satisfies the inequalities, for  $x > 0$ ,<sup>18)</sup>

$$\frac{1}{2} \ln(1 + 2/x) < e^x E_1(x) < \ln(1 + 1/x), (1+x)^{-1} < e^x E_1(x) < x^{-1}. \quad (19)$$

Accordingly, Eq. (17) gives formally for small and large arguments  $x = \Lambda^{-1}$ ,

$$L \approx \ln \Lambda, \quad \Lambda \gg 1; \quad L \approx \Lambda, \quad \Lambda \ll 1. \quad (20)$$

Rewriting  $\Lambda$  in terms of the thermal and quantum potential energies shows that for classical plasmas

$$\Lambda = 8E_T/E_Q \gg 1, \quad E_T \equiv KT, \quad E_Q \equiv \hbar^2/m\delta^2. \quad (21)$$

Combining of Eqs. (14)-(17) with Eq.(1) yields the desired electric conductivity of the classical plasma of intermediate nonideality,  $0.1 < \gamma \leq 1$ :

$$\sigma = 3(KT)^{3/2} / 2(2\pi m)^{1/2} e^2 ZL \quad (22)$$

where

$$L = \ln[8\pi\hbar^{-2} \alpha^{+2} (4\pi n/3Z)^{-2/3} KT], \quad \Lambda \gg 1, \quad (23)$$

by Eqs.(17) and (20).

The conductivity formula (22) differs from the conductivity of the ideal plasma<sup>1-3)</sup> mainly through the term  $L$ . The latter has the form of a Coulomb logarithm,  $L = \ln \Lambda$  for  $\Lambda \gg 1$ , i.e. for all densities  $n$  and temperatures  $T$  for which the plasma is nondegenerate,  $E_T > E_Q$ , Eq.(21). Numerically,

$$\Lambda = 3.482 \times 10^{11} \alpha^{+2} (n/Z)^{-2/3} T. \quad (24)$$

The corresponding argument  $\Lambda_D = 2KT D/Ze^2$  of the ideal Coulomb logarithm<sup>1-3)</sup>  $\ln \Lambda_D$ , is

$$\Lambda_D = 1.464 \times 10^4 Z^{-1} (1+Z)^{1/2} n^{-1/2} T^{3/2}. \quad (25)$$

Table I compares  $\Lambda$  of the nonideal plasma and  $\Lambda_D$  of the ideal plasma for large densities  $n$  and the typical temperature  $T = 10^4$  °K. It is seen that  $\ln \Lambda_D$  of the ideal plasma is unacceptably small for densities  $n \geq 10^{18} \text{ cm}^{-3}$ , whereas  $\ln \Lambda$  of the nonideal plasma has reasonable values up to densities  $n \leq 10^{22} \text{ cm}^{-3}$ , if  $T = 10^4$  °K. The conductivity formula (22) holds, therefore, for densities  $n$  up to 4 orders of magnitude higher than the conductivity formula of the ideal plasma. Eq. (22) is not applicable to  $n - T$  regions for which  $\Lambda \ll 1$ , i.e.  $E_T \ll E_Q$ , which would imply degenerate electrons.

TABLE I:  $\Lambda$  and  $\Lambda_D$  versus  $n$  for  $T = 10^4$  °K,  $Z = 1$ , and  $\alpha \sim 1$ .

$n[\text{cm}^{-3}]$	$10^{18}$	$10^{20}$	$10^{22}$	$10^{24}$
$\Lambda$	$3.482 \times 10^3$	$1.616 \times 10^2$	$0.750 \times 10^1$	$0.348 \times 10^0$
$\Lambda_D$	$1.035 \times 10^1$	$1.035 \times 10^0$	$1.035 \times 10^{-1}$	$1.035 \times 10^{-2}$

The conductivity formula (22) becomes in cgs - units or practical units ( $9 \times 10^{11} \text{ sec}^{-1} = 1 \text{ mho cm}^{-1}$ ),

$$\sigma = 1.394 \times 10^8 T^{3/2} / Z \ln \Lambda [\text{sec}^{-1}] = 1.549 \times 10^{-4} T^{3/2} / Z \ln \Lambda [\text{mho cm}^{-1}] \quad (26)$$

where  $\Lambda$  is given in Eq. (23). Accordingly, if  $T = 10^4$  °K and  $Z = 1$ ,  $\sigma = 1.899 \times 10^1 \text{ mho cm}^{-1}$  for  $n = 10^{18} \text{ cm}^{-3}$  and  $\sigma = 3.046 \times 10^1 \text{ mho cm}^{-1}$  for  $n = 10^{20} \text{ cm}^{-3}$ .

In Fig. 1, isobars of the dimensionless conductivity  $\sigma^* = m^{1/2} e^2 \sigma / (KT)^{3/2}$  are reproduced versus the number of electrons in the Debye sphere,  $N_D = 4\pi D^3 n / 3 \sim \gamma^{-3/2}$ , showing (1) conductivities according to the ideal plasma theory<sup>4</sup>, (2) computer conductivities from molecular dynamics methods (with error estimates)<sup>15</sup> and experimental conductivities<sup>16</sup> for (3) Cs and (4) Li plasmas at  $p = 5 \times 10^4 \text{ Pa}$ . Conductivity curves based on Eq. (22) are shown for (5) Li and (6) Cs plasmas at  $p = 5 \times 10^4 \text{ Pa}$  for comparison. It is seen that Eq. (22) is in good agreement with the machine calculations<sup>15</sup> and in reasonable accord with the experimental data<sup>16</sup>.

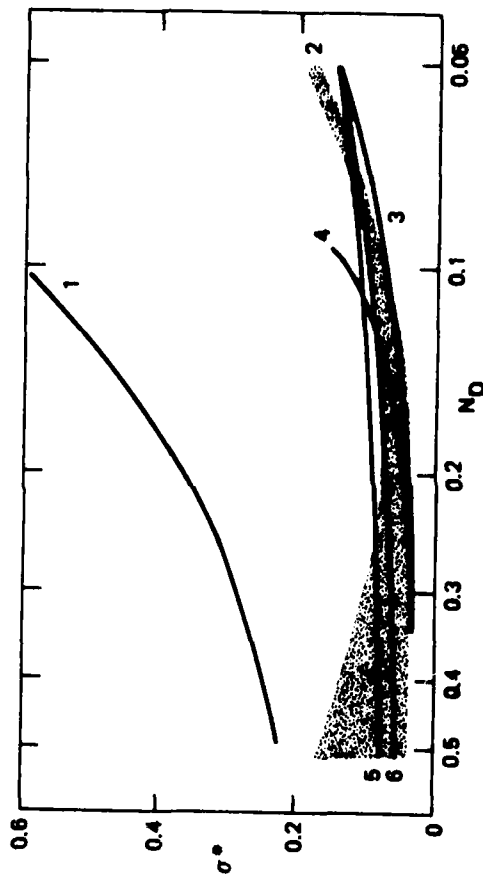


FIG. 1: Dimensionless conductivity  $\sigma^*$  versus number  $N_D$  of particles in the Debye sphere: (1) ideal plasma theory<sup>4</sup>, (2) molecular dynamics machine calculations<sup>15</sup>, experimental data<sup>16</sup> for (3) Cs and (4) Li plasmas at  $p = 5 \times 10^4$  Pa, and present theory for (5) Li and (6) Cs plasmas ( $p = 5 \times 10^4$  Pa).

# CONDUCTIVITY OF QUANTUM PLASMA

The electrons in a plasma become degenerate if their thermal DeBroglie wave length is larger than the mean electron distance, i.e. at densities

$$n > 4.828 \times 10^{15} T^{3/2} .$$

E.g., for  $T = 10^4$  °K, degeneracy requires  $n > 5 \times 10^{21} \text{ cm}^{-3}$ . In view of their large mass  $m_i \gg m$ , the ions can be treated as classical. The momentum relaxation time  $\tau$  of the degenerate electron gas is determined by the quantum statistical collision integral for  $m\vec{v}_e$ ,<sup>19)</sup>

$$\begin{aligned} -nm(\langle \vec{v}_e \rangle - \langle \vec{v}_i \rangle)/\tau = \\ m \int \dots \int \vec{v}_e \{ f_e(\vec{v}_e^*) f_i(\vec{v}_i^*) [1 - \frac{1}{2} \frac{h^3}{m} f_e(\vec{v}_e)] - f_e(\vec{v}_e) f_i(\vec{v}_i) [1 - \frac{1}{2} \frac{h^3}{m} f_e(\vec{v}_e^*)] \\ \times g \sigma(\theta, g) d\Omega d\vec{v}_e d\vec{v}_i \end{aligned} \quad (27)$$

where the scattering cross section  $\sigma(\theta, g)$  between electrons and ions is given by Eq. (11). The solutions to the velocity distributions are the displaced Maxwellian (13) for the ions ( $s = i$ ) and the 5-moment Fermi approximation for the electrons,

$$f_e(\vec{v}_e) = 2(m/h)^3 \{ 1 + \exp[\frac{1}{2}m(\vec{v}_e - \langle \vec{v}_e \rangle)^2 - \mu]/KT \}^{-1} . \quad (28)$$

The chemical potential  $\mu = \mu(n, T)$  is determined by the integral functional

$$n = \int f_e(\vec{c}_e; \mu) d\vec{c}_e .$$

Again, a linear response  $\vec{j} = \sigma \vec{E}$  exists for small drift velocities  $\langle \vec{v}_{e,i} \rangle$  or weak electric fields  $\vec{E}$ . Integration of Eq.(27) yields, after standard approximations, for the relaxation time of the degenerate electron gas:

$$\tau^{-1} = \frac{8}{3} \left( \frac{2}{m} \right)^{1/2} \frac{e^4 Z^2 n_i L_Q}{(KT)^{3/2} R(n, T)} \quad (29)$$

where

$$L_Q = e^{\Lambda_Q^{-1}} E_1(\Lambda_Q^{-1}), \quad (30)$$

$$\Lambda_Q = \frac{16}{3} \frac{m}{h^2} e^{+2} \left( \frac{4\pi n}{3Z} \right)^{-2/3} KT Q(n, T) \quad (31)$$

and

$$Q(n, T) = \frac{3}{2} (1 + 2^{-5/2} \frac{n}{\tilde{n}} + \dots), \quad n < \tilde{n}(T), \quad (32)$$

$$R(n, T) = \frac{4}{\pi^{1/2}} (1 + 2^{-2} \frac{n}{\tilde{n}} + \dots), \quad n < \tilde{n}(T), \quad (33)$$

but

$$Q(n, T) = \frac{3}{5} (\frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}})^{2/3} [1 + \frac{5\pi^2}{12} (\frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}})^{-4/3} + \dots], \quad n > \tilde{n}(T), \quad (34)$$

$$R(n, T) = \frac{1}{2} (\frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}})^1 [1 + \frac{3\pi^2}{4} (\frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}})^{-4/3} + \dots], \quad n > \tilde{n}(T). \quad (35)$$

Equations (32) - (33) and Eqs. (34) - (35) result from expansions of the Fermi distribution (28) in the collision integral (27) for densities  $n < \tilde{n}(T)$  and  $n > \tilde{n}(T)$ , respectively. Eqs. (32) and (34) indicate that  $\Lambda_Q \gg 1$  for  $n \ll \tilde{n}$  and  $\Lambda_Q \ll 1$  for  $n \gg \tilde{n}$  since  $\Lambda_Q = (2/3)(\beta/\alpha)^2 \Lambda Q(n, T)$  by Eq. (31).

The series are based on expansions of the normalization integral in Eq. (28), which gives the chemical potential  $\mu$  explicitly as a function of  $n$  and  $T$ ,

$$\frac{\mu}{kT} = \ln \left\{ \frac{n}{\tilde{n}} \left[ 1 + 2^{-3/2} \left( \frac{n}{\tilde{n}} \right)^1 + \left( \frac{1}{4} - 3^{-3/2} \right) \left( \frac{n}{\tilde{n}} \right)^2 + \dots \right] \right\}, \quad n < \tilde{n}(T), \quad (36)$$

$$\frac{\mu}{kT} = \left( \frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}} \right)^{2/3} \left[ 1 - \frac{\pi^2}{12} \left( \frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}} \right)^{-4/3} - \frac{\pi^4}{80} \left( \frac{3\pi^{1/2}}{4} \frac{n}{\tilde{n}} \right)^{-8/3} + \dots \right], \quad n > \tilde{n}(T) \quad (37)$$

Combining of the conductivity formula in Eq. (1) with the relaxation time of Eq. (29) yields for the electrical conductivity of the degenerate electron plasma of intermediate nonideality,  $0.1 < \Upsilon \leq 1$ :

$$\sigma = \frac{3(kT)^{3/2} R(n, T)}{8(2m)^{1/2} e^2 Z L_Q} \quad (38)$$

where  $L_Q$  is given by Eq. (30). In the limiting cases of large and small values of  $\Lambda_Q$ ,

$$L_Q = \ln \left[ \frac{2}{3} \left( \frac{\beta}{\alpha} \right)^2 \Lambda Q(n, T) \right], \quad \Lambda_Q \gg 1, \quad (39)$$

$$L_Q = \frac{2}{3} \left( \frac{\beta}{\alpha} \right)^2 \Lambda Q(n, T), \quad \Lambda_Q \ll 1, \quad (40)$$

since  $\Lambda_Q = (2/3)(\beta/\alpha)^2 \Lambda_Q$  by comparison of Eqs.(31) and (17).

For  $n/\tilde{n} \rightarrow 0$ , Eqs.(38) and (39) reduce to the classical conductivity, Eqs.(22) and (23), since  $\mathcal{R}(n,T) \rightarrow 4/\sqrt{\pi}$  and  $\mathcal{Q}(n,T) \rightarrow 3/2$  for  $n/\tilde{n} \rightarrow 0$  by Eqs.(32) - (33). On the other hand, Eqs.(38) and (40) give in the limit of complete degeneracy,  $n/\tilde{n} \rightarrow \infty$ :

$$\sigma = \frac{9h^3 n}{2^9 \pi m^2 e^2 Z L_Q} \quad (41)$$

where

$$\Lambda_Q = \frac{12}{5} (3\pi^2 Z^2/2)^{1/3} \beta^2 \quad (42)$$

by Eqs.(34) and (35). Since  $M = \frac{1}{2}$  and  $N = 0$  for  $n/\tilde{n} \rightarrow \infty$ ,

$$\beta^2 = [Z/4\pi(1+Z)](4\pi/3Z)^{2/3} \Gamma^{-1} \quad (43)$$

by Eq.(4), i.e.,  $\beta \geq 1$  depending on the magnitude of  $\Gamma = E_c/E_Q$ , Eq.(8).

Equation (43) agrees with the expression for the conductivity of a low temperature metal.<sup>20)</sup>

## GENERALIZATION

Nonideal plasmas may exhibit not only a high degree of single ionization but also multiple ionization, due to lowering of the ionization energies by the internal Coulomb fields, and overlapping of the atomic wave functions at sufficiently high pressures.<sup>21)</sup> In an electrically neutral plasma with  $N$  species of ions ( $i$ ) of charge  $Z_i e$  and density  $n_i$ , the electron density  $n$  and entire ion density  $n(i)$  are related by

$$n = \sum_{i=1}^N Z_i n_i, \quad n(i) = \sum_{i=1}^N n_i. \quad (44)$$

Since the probabilities  $\tau_i^{-1}$  for interaction between the electrons and ions of type  $i = 1, 2, \dots, N$  are additive, the momentum relaxation time of the electrons is in presence of  $N$  ion components ( $i$ ) given by

$$\tau^{-1} = \sum_{i=1}^N \tau_i^{-1}. \quad (45)$$

From Eqs. (44) to (45) follows that the derived conductivity formulas are generalized to many-ion-component plasmas by means of the substitutions:

$$Z \rightarrow \bar{Z} = \sum_{i=1}^N n_i Z_i / n(i), \quad (46)$$

$$Z^2 \rightarrow \overline{Z^2} = \sum_{i=1}^N n_i Z_i^2 / n(i), \quad (47)$$

$$Z^{-1} \rightarrow \overline{Z^{-1}} = \bar{Z} / \overline{Z^2}. \quad (48)$$

Since it is extremely difficult to calculate accurately the ion densities  $n_i$  in many-component nonideal plasmas,<sup>22)</sup> it is advisable to make use of equivalent approximations [see, e.g., Eq. (9)] to avoid too cumbersome conductivity expressions in practical applications.

## APPENDIX

Since adequate methods for the solution of the many-body problem are not available, we derive the shielding radius  $\delta$  of nonideal plasmas from first principles of dimensional theory<sup>23)</sup>. This method gives  $\delta$  in dependence of the relevant dimensional plasma parameters ( $e$ ,  $m$ ,  $n$ ,  $KT$ ,  $\hbar$ ), but leaves a dimensionless proportionality constant of order one undetermined,  $C_0 \sim 1$ .

The latter can, however be found from the physical argument

$$\delta = D \equiv [KT/4\pi(1+Z)e^2n]^{1/2} \quad \text{for } \gamma \ll 1, \quad n \ll \tilde{n}, \quad (\text{A1})$$

since  $\delta$  approaches the Debye radius  $D$  in the limit of the weakly nonideal classical plasma. The dimensions of the characteristic plasma quantities, which determine  $\delta$  of dimension  $L$  are given in terms of the fundamental dimensions of length ( $L$ ), time ( $T$ ), and mass ( $M$ ):

$$D[e] = L^{3/2} M^{1/2} T^{-1}, \quad D[m] = M, \quad D[n] = L^{-3}, \quad D[KT] = ML^2 T^{-2}, \quad D[\hbar] = ML^2 T^{-1}. \quad (\text{A2})$$

A. Classical Plasma. Since in a classical plasma  $\delta$  depends on the dimensional parameters  $e$ ,  $m$ ,  $n$ , and  $KT$ , Dupré's fundamental theorem of dimensional analysis<sup>23)</sup> demonstrates that

$$\delta = C_0 e^{N_1} m^{N_2} n^{N_3} (KT)^{N_4} \quad (\text{A3})$$

where

$$\frac{3}{2}N_1 - 3N_3 + 2N_4 = 1, \quad \frac{1}{2}N_1 + N_2 + N_4 = 0, \quad -N_1 - 2N_4 = 0, \quad (\text{A4})$$

by comparison of the powers  $N_i$  of  $L$ ,  $M$ , and  $T$  in Eq. (A3). Elimination of

$N_1$ ,  $N_2$ ,  $N_3$  in terms of  $N_4 = N$  reduces Eq. (A3) to

$$\delta = C_0 (e^2 n^{1/3} / KT)^{-N} n^{-1/3} \quad (\text{A5})$$

It is seen that  $\delta = C_0 n^{-1/3}$  for  $N = 0$  (strongly nonideal plasma,  $\gamma \gg 1$ , with negligible thermal energy) and  $\delta = D$  for  $N = 1/2$  (weakly nonideal plasma,  $\gamma \ll 1$ ).

Accordingly,

$$C_0 = [4\pi(1+Z)]^{-1/2} \quad (\text{A6})$$

by Eq. (A1), Eqs.(A5) and (A6) are combined in an illustrative form for classical plasmas:

$$\delta = \alpha_0 \left( \frac{Ze^2 n^{1/3}}{KT} \right)^{-N} \left( \frac{3Z}{4\pi n} \right)^{1/3}, \quad 0 \leq N \leq 1/2 \quad (A7)$$

This is Eq. (2) where  $\alpha_0(Z)$  is defined in Eq. (6).

B. Quantum Plasma. Since in a quantum plasma  $\delta$  depends on the dimensional parameters  $e, m, n, KT$ , and  $\hbar$ , Dupré's theorem<sup>23)</sup> gives

$$\delta = C_0 e^{N_1} m^{N_2} n^{N_3} (KT)^{N_4} \hbar^{N_5} \quad (A8)$$

where

$$\frac{3}{2}N_1 - 3N_3 + 2N_4 + 2N_5 = 1, \quad \frac{1}{2}N_1 + N_2 + N_4 + N_5 = 0, \quad -N_1 - 2N_4 - N_5 = 0 \quad (A9)$$

by comparison of the powers  $N_i$  of  $L, M$ , and  $T$ , in Eq. (A8). Elimination of  $N_1$ ,

$N_2, N_3$  in terms of  $N_4 = N$  and  $N_5 = 2M$  reduces Eq. (A8) to

$$\delta = C_0 \left( \frac{e^2 n^{1/3}}{KT} \right)^{-N} \left( \frac{e^2 n^{1/3}}{\hbar^2 m^{-1} n^{2/3}} \right)^{-M} n^{-1/3} \quad (A10)$$

It should be noted that Eqs. (A5) and (A7) are the special case  $N_5 = 0$  of Eq. (A8) and  $M = 0$  of Eq. (A10), respectively. For  $M = 1/2$ , Eq. (A10) reduces to the Fermi shielding length<sup>13)</sup> of the completely degenerate plasma ( $n \gg \tilde{n}$ ).

Eqs. (A6) and (A10) are rewritten in an illustrative form for quantum plasmas:

$$\delta = \beta_0 \left( \frac{Ze^2 n^{1/3}}{KT} \right)^{-N} \left( \frac{Ze^2 n^{1/3}}{\hbar^2 m^{-1} n^{2/3}} \right)^{-M} \left( \frac{3Z}{4\pi n} \right)^{1/3}, \quad 0 \leq N, M \leq \frac{1}{2} \quad (A11)$$

This is Eq. (4) where  $\beta_0$  is defined in Eq. (6).

In the above formulas, the power  $0 \leq N \leq 1/2$  characterizes the nonideality ( $N = 1/2$  for  $\gamma \ll 1$ ,  $N = 0$  for  $\gamma \gg 1$ ), whereas the power  $0 \leq M \leq 1/2$  characterizes the degeneracy ( $M = 0$  for  $n \ll \tilde{n}$ ;  $M = 1/2$  for  $n \gg \tilde{n}$ ). Although dimensional theory alone does not provide expressions for  $M$  and  $N$ , it is recognized that  $\alpha(\gamma)$  [Eq. (3)] and  $\beta(\gamma, \Gamma)$  [Eq. (5)] are of magnitude-of-order one, i.e.  $\delta \sim (3Z/4\pi n)^{1/3}$  for nonideal classical and quantum plasmas of intermediate nonideality,  $0.1 < \gamma \leq 1$ .

In the conductivity theory of metals<sup>20,24,25)</sup>, the mean ion radius  $r_i = (3Z/4\pi n)^{1/3}$  is used widely as shielding length of the ion potential, based on phenomenological arguments. The presented dimensional analysis provides the first mathematical justification not only for nonideal plasmas but also for metals.

# REFERENCES

- \* ) Supported by the US Office of Naval Research.
- 1. R. Landshoff, Phys. Rev. 76, 904 (1949).
- 2. R. S. Cohen, L. Spitzer, and P. M. Routhly, Phys. Rev. 80, 230 (1950).
- 3. H. E. Wilhelm, Can. J. Phys. 51, 2604 (1973).
- 4. L. Spitzer, Physics of Fully Ionized Gases (Interscience, New York 1956).
- 5. S. G. Barolskii, N. V. Ermokhin, P. P. Kulik, and V. M. Melnikov, Sov. Phys. JETP 35, 94 (1972).
- 6. Yu. V. Ivanov, V. B. Mintsev, V. E. Fortov, and A. N. Dremin, Sov. Phys. - JETP 44, 112(1976).
- 7. C. Goldbach, G. Nollez, S. Popovic, and M. Popovic, Z. Naturforsch. 33a, 11 (1977).
- 8. N. N. Iermohin, B. M. Kovaliov, P. P. Kulik, and V. A. Riabii, J. Physique 39, Supp. 5, 200(1978).
- 9. H. E. Wilhelm, Phys. Rev. 187, 382(1969).
- 10. V. S. Rogov, Teplofiz. Vys. Temp. 8, 689(1970).
- 11. Ref. 7, p. 16, Eq.(11).
- 12. Y. L. Klimontovich, Sov. Phys.- JETP 35, 920(1972).
- 13. N. F. Mott, Proc. Camb. Phil. Soc. 32, 281(1936).
- 14. T. Wu and T. Ohmura, Quantum Theory of Scattering (Prentice-Hall, Englewood-Cliffs 1962).
- 15. A. A. Valuev and G. E. Norman, High Temp. 15, 689 (1975).
- 16. I. Ya. Dikhter, V.A.Zeigarnik, and S.V.Smagin, High Temp. 17, 256 (1979).
- 17. See Ref. 9, Eqs. (37) - (41); the relaxation time for supersonic drift velocities is given in Eqs. (27) - (30).
- 18. M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (Dover, New York 1965).
- 19. E. A. Uehling and G. E. Uhlenbeck, Phys. Rev. 43, 552 (1933).
- 20. A. Sommerfeld and H. Bethe, Electron Theory of Metals, Handbook of Physics, Vol. 24/2 (Springer, Berlin 1933).

21. H. E. Wilhelm, IEEE Tr. Plasma Science PS-8,9 (1980).
22. W. Ebeling, C. V. Meister, and R. Sandig, Ann. Physik, 36, 321,(1979).
23. A. W. Porter, The Method of Dimensions (Methuen, London 1946).
24. A. Haug, Theoretical Solid State Physics I, II (Pergamon, New York, 1972).
25. R. Kubo and T. Nagamiya, Solid State Physics (McGraw-Hill, New York 1969).

# ELECTRICAL CONDUCTIVITY OF NONIDEAL QUASI-METALLIC PLASMAS

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## Abstract

Electrical conductivity formulas are derived from first principles for fully ionized nonideal plasmas. The theory is applicable to an electron-ion system with a i) Maxwell electron distribution with an arbitrary interaction parameter  $\gamma = Ze^2 n^{1/3} / kT$  (ratio of the mean Coulomb interaction and thermal energies) and ii) Fermi electron distribution with an interaction parameter  $\Gamma = Ze^2 n^{1/3} / \hbar^2 m n^{2/3}$  (ratio of the Coulomb interaction and Fermi energies). The momentum relaxation time of the electrons in the plasma is calculated based on plane electron wave functions interacting with the continuum oscillations (plasma waves) through a shielded Coulomb potential  $U_s(r) = e_s e \exp(-r/\delta_s) / r$ , which takes into account both electron-ion interactions ( $s=i$ ) and electron-electron interactions ( $s=e$ ). It is shown that the resulting conductivity formulas are applicable to higher densities, for which the ideal plasma conductivity theory breaks down because the Debye radius loses its physical meaning as a shielding length and upper impact parameter. The conductivity obtained for classical plasma is of the form  $\sigma_c = \sigma_c^* (kT)^{3/2} / m^{1/2} e^2$  and agrees with the ideal plasma conductivity formula with respect to the temperature and density dependence for  $\gamma/Z \rightarrow 0$ , but its magnitude is significantly reduced as  $\gamma/Z$  increases. For quantum plasmas, the conductivity obtained is of the form  $\sigma_Q = \sigma_Q^* \hbar^3 n / m^2 Ze^2$ , which shows that the degenerate plasma behaves like a low temperature metal.

# 1. INTRODUCTION

The electrical conductivity of nonideal plasmas has been subject to many experimental and theoretical investigations<sup>1-7)</sup>. The theory of the electrical conductivity of fully ionized plasmas, based on the Boltzmann equation,<sup>8)</sup> the Fokker-Planck equation<sup>9)</sup> (derived by expanding the binary collision integral for the small, successive velocity changes of coulomb scattering), or on the Lenard-Balescu equation<sup>10)</sup> (taking into account the dielectric properties of the medium) is in agreement with the experimental data only for rarefied high temperature plasmas,  $\gamma/Z \ll 1$ . The interaction parameter  $\gamma/Z$  is defined as the ratio of (average) Coulomb interaction ( $Ze^2 n^{1/3}$ ) and thermal ( $KT$ ) energies ( $n$  is the electron density and  $Z$  the ion charge number),

$$\gamma = Ze^2 n^{1/3} / KT = 1.670 \times 10^{-3} Z n^{1/3} / T$$

(cgs-units are used throughout). The conventional transport calculations for classical ideal plasmas<sup>8-10)</sup> and weakly nonideal plasmas<sup>1-2)</sup> give an electrical conductivity of the form  $\sigma \sim (kT)^{3/2} / m^{1/2} e^2 Z \ln \Lambda_D$ , where  $\Lambda_D = [1 + (d_D/p_0)^2]^{1/2} \approx d_D/p_0$  for  $d_D \gg p_0$ . The impact parameter for 90° deflections (Landau length) is  $p_0 = Ze^2 / 3KT$ . The condition,  $\Lambda_D \gg 1$  or  $\ln \Lambda_D \sim 10^1$  is satisfied only for not too low temperatures  $T$  and not too high densities  $n$ . Conductivity formulas with this Coulomb logarithm break down for large interactions parameters  $\gamma/Z$  and densities  $n$ , since the Debye radius

$$d_D = [Z / 4\pi(1+Z)]^{1/2} \gamma^{-1/2} n^{-1/3}$$

becomes of the order and smaller than the mean particle distance  $n^{-1/3}$  for  $\gamma/Z \sim 10^{-1}$  and  $\gamma/Z > 10^{-1}$ , respectively. Thus  $d_D$  loses its physical meaning as an electrical shielding length and maximum impact parameter. For this reason, a new shielding length  $\delta$  is introduced through dimensional analysis<sup>17)</sup>.

Moderately nonideal plasmas with  $\gamma/Z \sim 1$  are readily generated through shock wave compression and exhibit conductivities of the order  $\sigma \sim 10^1 - 10^2 \text{ mho/cm}$ <sup>12-13)</sup>, which are much smaller than those which would be obtained by applying the

conductivity formula for ideal plasmas in the nonideal regime. Although there are bulk measurements of the electrical conductivity of nonideal Cesium and noble gases plasmas available<sup>12-15)</sup>, the formulas are valid only for small interaction parameters<sup>1-2)</sup>  $\gamma/Z$ . The momentum and energy transport in weakly nonideal plasmas ( $\gamma/Z \ll 1$ ) was treated by Wilhelm<sup>16)</sup> by means of an exponentially shielded Coulomb potential<sup>1</sup>, which permits to consider not only short-range binary ( $r \leq d_D$ ) but also long-range many-body ( $r > d_D$ ) interactions. This interaction model was used shortly afterwards by Rogov<sup>2)</sup> for the calculation of the conductivity of weakly nonideal Argon and Xenon plasmas with Debye shielding. Later, Wilhelm<sup>17)</sup> applied his theory to nonideal plasmas by deriving a shielding length and Coulomb logarithm which are valid for  $0 < \gamma/Z < 1$ .

For moderately nonideal plasmas ( $0.1 < \gamma/Z \leq 1$ ) various phenomenological approaches have been used to extend the conductivity formulas of ideal plasmas, e.g., Goldbach et al<sup>4)</sup> multiply the Debye length  $d_D$  with a free parameter  $x(p)$  which is chosen to match the experimental data, i.e. to compensate for the too rapid decrease of  $d_D$  with pressure. A kinetic equation has been proposed for nonideal plasmas by Klimontovich<sup>19)</sup>, which considers spatial correlations and temporal retardation in the collision integrals, but does not take into account many-particle collisions. Ebeling et al<sup>1)</sup> used recently kinetic<sup>20)</sup> and correlation function<sup>21)</sup> methods to derive a resistance formula for nonideal plasmas, which is applicable only for  $\gamma/Z \ll 1$ .

Herein, we extend and apply the Bloch<sup>28)</sup> transport theory to nonideal plasmas, based on concepts similar to those used for solids<sup>23)</sup> and liquid metals<sup>24,25)</sup>. The application of this model to nonideal plasmas is justified since a plasma exhibits a quasi-crystalline structure for  $\gamma/Z > 10^{-1}$ , becomes a liquid for  $\gamma/Z \sim 1$ , and undergoes a diffuse transition into a solid, metallic state at a critical value  $\gamma_0$ . The role of the longitudinal phonons in the

theory of metals is assumed by the quanta of the plasma oscillations (plasmons).

The theory to be presented provides a momentum relaxation time for

i) classical plasmas ( $n < \tilde{n}$ ,  $\tilde{n} = 2(2\pi m k T)^{3/2} / \hbar^3$ ) with an interaction parameter  $\gamma/Z$  and ii) quantum plasma ( $n > \tilde{n}$ ) with an interaction parameter  $\Gamma/Z$  ( $\Gamma = Ze^2 n^{1/3} / \hbar^2 m^{-1} n^{2/3}$ ). The results are compared with previous theories and experiments.

## II. ELECTRON-PLASMON INTERACTION

We consider the electrical conductivity of nonideal plasmas due to the many-body interactions of the electrons with longitudinal plasma waves (similar to the interaction of electrons with phonons in liquids or solids). The plasma under consideration is a continuum of volume  $\Omega$  containing  $N$  electrons and  $N/Z$  ions, which exhibits  $3N$  (high frequency branch) and  $3N/Z$  (low frequency branch) characteristic frequencies  $\omega_s(q)$  of longitudinal oscillations ( $s=e,i$ ). The high frequency branch corresponds to electron plasma oscillations and the low frequency branch are the ion sound waves.

The motion of electrons in a continuum is affected by the continuum oscillations (many-body interactions). In ideal plasmas, the change in motion is caused by binary collisions of the electrons with the plasma particles. In nonideal plasmas, however, the electrons interact with the fluctuating Coulomb field of all charged particles. Therefore, this interaction can be treated as a scattering of the electrons by the random longitudinal waves of the plasma continuum, which are thermally excited.

As in the theory of metals<sup>24, 25)</sup> we are considering a free electron model, which is applicable to nonideal plasmas. For dense plasmas with  $Z$  electrons per ion, the electron wave functions are approximated by plane waves  $\sim \exp(i\vec{k} \cdot \vec{r})$ . The electron energy  $E$  is given in terms of the wave vector  $\vec{k}$  by  $E = \hbar^2 k^2 / 2m$ , so that the Fermi surface is spherical.

Let  $\omega_e(\vec{q})$  be the  $e^{\text{th}}$  eigenoscillation with wave vector  $\vec{q}$  of an electron wave (e-plasmon) and,  $\omega_i(\vec{q})$  the  $i^{\text{th}}$  eigenoscillation with wave vector  $\vec{q}$  of an ion

sound wave (i-plasmon). Taking into consideration conservation of energy and momentum, i.e.  $\hbar\omega(\vec{k}) = E' - E$  and  $\hbar\vec{q} = \vec{p}' - \vec{p}$ , where  $E'$ ,  $\vec{p}'$  and  $E$ ,  $\vec{p}$  are respectively the energy and momentum of an electron before and after a collision with a plasmon of energy  $\hbar\omega(\vec{q})$  and momentum  $\hbar\vec{q}$ , we see that an electron or ion interacting with the plasma as a whole can emit and absorb plasmons which are quasi-particles with energy  $\hbar\omega(\vec{q})$  and momentum  $\hbar\vec{q}$ .

The quasi-particles or plasmons obey Bose-Einstein statistics, and their distribution function is

$$\bar{N}_q = \frac{1}{\exp\left[\frac{\hbar\omega(q)}{KT}\right] - 1} \quad (1)$$

Let  $P(\vec{k}, \vec{k}')$  be the transition probability per unit time that upon a collision of the electron with a plasmon, an electron in a state  $\vec{k}$  moves to another state  $\vec{k}'$  which is not occupied by any other electron. If  $f(\vec{k})$  is the distribution function of the electron occupying the state  $\vec{k}$  and  $f(\vec{k}')$  the distribution function of the electron in the state  $\vec{k}'$ , the number of electrons which move from the state  $\vec{k}$  to the state  $\vec{k}'$  is (Pauli principle)

$$P(\vec{k}, \vec{k}')f(\vec{k})[1 - f(\vec{k}')] \quad .$$

Since there exists always an inverse transition to the above forward interaction, the total rate of change in time of  $f(\vec{k})$  due to electron-wave interactions is obtained by summing over all  $\vec{k}'$ ,

$$\frac{\delta f(\vec{k})}{\delta t} = \sum_{\vec{k}'} \{P(\vec{k}', \vec{k})f(\vec{k}') [1 - f(\vec{k})] - P(\vec{k}, \vec{k}')f(\vec{k}) [1 - f(\vec{k}')] \} \quad (2)$$

The interaction processes are calculated by perturbation theory. According to Akhiezer<sup>26)</sup> or Schiff<sup>27)</sup>, the probability of transition from an initial state  $\vec{k}$  to a final state  $\vec{k}'$  is

$$P(\vec{k}, \vec{k}') = \frac{2\pi}{\hbar} |M_{kk'}|^2 \delta(E_{k'} - E_k) \quad (3)$$

$E_{k'}$  and  $E_k$  are the energies of the electron in the states  $\vec{k}'$  and  $\vec{k}$ , respectively.

$|M_{kk'}|$  is the matrix element of the transition  $\vec{k} \rightarrow \vec{k}'$ . For the absorption of a plasmon,  $|M_{kk'}|$  is proportional to  $\bar{N}_q$ , and for the emission of a plasmon it is proportional to

$(\bar{N}_q + 1)$ . The more remarkable dependence of  $|M_{kk'}|$  is on the Fourier transform of the potential  $U_s(r)$ , by means of which the particles in the plasma interact. Similar to the matrix element of a metal given by Sham and Ziman<sup>28)</sup>,  $|M_{kk'}|$  for electron ( $s=e$ ) and ion ( $s=i$ ) oscillations with different frequency  $\omega_s$  is

$$|M_{kk'}|^2 = |\alpha_q^s|^2 (\vec{q} \cdot \vec{e}_q)^2 |U_s(q)|^2, \quad s=e, i, \quad (4)$$

where

$$|\alpha_q^s|^2 = \frac{\hbar \bar{N}_q^s}{2m_s n_s \omega_s(\vec{q})} \quad \text{for the absorption of a plasmon}, \quad (5)$$

$$|\alpha_q^s|^2 = \frac{\hbar(\bar{N}_q^s + 1)}{2m_s n_s \omega_s(\vec{q})} \quad \text{for the emission of a plasmon}.$$

For plane waves normalized in a unit volume, the Fourier transform of  $U_s(r)$

( $e_e = -e$ ,  $e_i = Ze$ ) is given by

$$|U_s(q)| = \frac{4\pi n_s e |e_s|}{\delta_s^{-2} + q^2}, \quad q = |\vec{k}' - \vec{k}|. \quad (6)$$

$m_s$  is the mass of the particle  $s$ ,  $n_s$  is its density,  $|\alpha_q^s|^2$  is the mean square amplitude of the  $q^{\text{th}}$  mode of an oscillation of frequency  $\omega_s(\vec{q})$ ,  $\vec{e}_q$  is the unit vector direction of the propagation vector  $\vec{q}$ , and  $|U_s(\vec{q})|^2$  is the square of the Fourier transform of the potential (through which the electrons and ions are interacting in the plasma).

Instead of using a phenomenological pseudo-potential<sup>1)</sup>  $U_s(r)$  with adjustable parameters we describe the nonideal plasma with classical ( $\gamma/Z$ ) and quantum mechanical ( $\Gamma/Z$ ) interaction parameters by means of a Yukawa potential with a shielding radius<sup>17)</sup>

$\delta_s^{-1/3} (n=n_e=N/\Omega, n_i=N/Z\Omega)$ :

$$U_s(r) = \frac{e_s e \exp(-r/\delta_s)}{r}, \quad s=e, i, \quad (7)$$

where

$$\delta_i = (4\pi n/3Z)^{-1/3} \{1 + (4\pi/3Z)^{1/3} [4\pi(1+Z)]^{-1/2} (\gamma/Z)^{-1/2}\} \quad (8)$$

with  $\delta_i = d_D$  for  $\gamma/Z \ll 1$  and  $\delta_i = (4\pi n/3Z)^{-1/3}$  for  $\gamma/Z \gg 1$ .

The plasma system consists of two components, electrons and ions where every charge ( $-e$  for  $s=e$  and  $Ze$  for  $s=i$ ) is assumed to be uniformly spread over a spherical cell of Wigner-Seitz type of radius  $\delta_s$  ( $s=e,i$ ), from which the electrons are scattered in accordance with the exponentially shielded potential  $U_s$ .  $\delta_i$  being defined above, a radius  $\delta_e$  is to be defined in order to take into account the electron-electron interaction.  $\delta_e$  is assumed to coincide with the shortest wavelength  $\lambda_e \approx 2\pi/\hat{q}_e$  for  $n > \tilde{n}$  and  $\Gamma/Z \geq 1$ , and  $\hat{q}_e$  is defined by the conservation of the total number of degrees of freedom of the electron gas, i.e.,

$$(4\pi)^{-3} \int_0^{\hat{q}_e} 4\pi q^2 dq = 3N. \quad (9)$$

For  $n < \tilde{n}$  and  $\Gamma/Z \leq 1$ , the minimum wavelength is obtained through the mean particle distance  $\lambda_e \sim \bar{r}_e$ , where  $\bar{r}_e = (4\pi n/3)^{-1/3}$ . Accordingly

$$\delta_e = 2\pi(18\pi^2 n)^{-1/3}, \quad n > \tilde{n}, \quad (10)$$

$$\delta_e = (4\pi n/3)^{-1/3}, \quad n < \tilde{n}, \quad (11)$$

1. Electron Oscillations. The high frequency branch of the space-charge waves is due to longitudinal electron oscillations. Their frequencies  $\omega_e(\vec{q})$  ( $s=e$  in Eq.(5)) are for classical ( $n < \tilde{n}$ ) and completely degenerate ( $n > \tilde{n}$ ) electrons

$$\omega_e(q) = \omega_p (1 + a^2 q^2)^{1/2} \quad (12)$$

$$a^2 = C_m^2 / \omega_p^2, \quad n < \tilde{n}, \quad (13)$$

$$a^2 = \frac{3}{5} v_F^2 / \omega_p^2, \quad n > \tilde{n}, \quad (14)$$

where the speed of sound  $C_m$  and the Fermi Speed  $v_F$  of the electrons are

$$C_m = (\kappa_e kT/m)^{1/2}, \quad v_F = \hbar(3\pi^2 n)^{1/3}/m, \quad (15)$$

and the critical electron density  $\tilde{n}$  and plasma frequency  $\omega_p$  are given by

$$\tilde{n} = 2(2\pi m kT/\hbar^2)^{3/2}, \quad (16)$$

$$\omega_p = (4\pi n e^2/m)^{1/2}, \quad (17)$$

( $\kappa_e = (C_p/C_v) = 5/3$  for the electrons and  $m$  is their mass).

For all wave numbers,  $0 \leq q \leq \hat{q}_e \sim 2\pi/\delta_e$ , the electron oscillations propagate with frequency  $\omega = \omega(q) > \omega_p$  in nonideal plasmas.

2. Ion Oscillations. The low frequency branch of the space charge waves is due to ions sound waves, which are coupled with the electrons. Since the ions are nondegenerate, the frequency of the ion oscillations is 31)

$$\omega_i(q) = v(q)C_s q, \quad C_s = \left( \frac{\kappa_i kT}{M} \right)^{1/2}, \quad (18)$$

where

$$v(q) = \left[ 1 + \frac{Z(\kappa_e/\kappa_i)}{1 + Z^{-1} \kappa_e(q\delta_e)^2 / (36\pi)^{1/3}} \right]^{1/2}, \quad n \ll \tilde{n}, \quad (19)$$

and

$$v(q) \approx 1, \quad n \gg \tilde{n}. \quad (20)$$

$v(q)$  is a correction factor of magnitude-of-order 1, which shows the influence of the electrons on the ion oscillation ( $M$ =ion mass,  $\kappa_i = C_p/C_v = 5/3$  for the ions) where  $0 \leq q \leq \hat{q}_i \sim 2\pi/\delta_i$ .

### III. RELAXATION TIME

The distribution function  $f(\vec{k})$  in Eq. (2) is not symmetric with respect to the origin in the  $\vec{k}$ -space<sup>23)</sup>, since it is "polarized" by the electric field  $\vec{E}$ . If  $\vec{E}$  is in the  $\vec{x}$ -direction,  $f(\vec{k})$  has the form

$$f(E) = f_0(E) + \phi, \quad \phi = -\frac{\partial f_0}{\partial E} \Phi, \quad -\frac{\partial f_0}{\partial E} = \frac{f_0(1-f_0)}{KT}. \quad (21)$$

The Fermi distribution function describing the thermal equilibrium of the electrons is

$$f_0(E) = [1 + e^{(E-\xi)/KT}]^{-1}, \quad (22)$$

where  $\xi$  is the Fermi energy, and  $\Phi$  is proportional to a function  $C(E)$  of the energy  $E$  of the electrons<sup>22)</sup>

$$\phi = e |\vec{E}| v_x C(E). \quad (23)$$

If equation (2) is changed from the discrete summation over  $\vec{k}'$  to an integral [where the volume  $\Omega$  is set to unity because the electron plane waves have been normalized for a unit volume in Eq. (6)] Eq. (2) becomes

$$\frac{\delta f}{\delta t} = \frac{1}{(2\pi)^3} \int \left\{ P(\vec{k}, \vec{k}') f(\vec{k}') [1-f(\vec{k})] - P(\vec{k}', \vec{k}) f(\vec{k}) [1-f(\vec{k}')] \right\} d^3 \vec{k}'. \quad (24)$$

The full expressions for  $P(\vec{k}, \vec{k}')$  and  $P(\vec{k}', \vec{k})$  are given by Eqs. (3) - (6). For absorption of a plasmon,

$$P(\vec{k}, \vec{k}') = \frac{\pi N^s q^2}{m_s n_s \omega_s(q)} |U_s(q)|^2 \delta(E' - E - \hbar \omega_s) \quad (25)$$

where

$$E' = E + \hbar \omega_s, \quad \vec{k}' = \vec{k} + \vec{q}. \quad (26)$$

For emission of a plasmon,

$$P(\vec{k}', \vec{k}) = \frac{\pi(\bar{N}\bar{q}+1)q^2}{m_S n_S \omega_S(q)} |U_S(q)|^2 \delta(E' - E + \hbar\omega_S) \quad (27)$$

where

$$E' = E - \hbar\omega_S, \quad \vec{k}' = \vec{k} - \vec{q} \quad (28)$$

Due to the fact that in equilibrium  $\frac{\delta f_0}{\delta t} = 0$  for a Fermi distribution function, it is easy to show that (detailed microscopic balance for direct and inverse interactions)

$$W(\vec{k}', \vec{k}) = P(\vec{k}, \vec{k}') f_0(\vec{k}') [1 - f_0(\vec{k})] = P(\vec{k}', \vec{k}) f_0(\vec{k}) [1 - f_0(\vec{k}')] = W(\vec{k}, \vec{k}') \quad (29)$$

Accordingly, the linearized interaction integral of the Boltzman equation is obtained by substituting Eq.(21) in Eq. (24), under consideration of the relation (29). Limiting ourselves to first-order terms in  $\phi$ , Eq. (24) becomes

$$\frac{\delta f}{\delta t} = \frac{1}{(2\pi)^2 K T} \int W(\vec{k}', \vec{k}) [\phi(\vec{k}') - \phi(\vec{k})] d^3 \vec{k}' \quad (30)$$

According to Haug<sup>22)</sup> a momentum relaxation time exists in a closed system of particles in presence of an electric field, and the interaction integral can be written as:

$$\frac{\delta f}{\delta t} = -\frac{f - f_0}{\tau} = -\frac{\phi}{\tau} \quad (31)$$

where the relaxation time  $\tau$  is in general a function of energy  $E$ . Hence

$$\tau(E) = -\frac{\phi}{\frac{\delta f}{\delta t}} \quad (32)$$

Eq.(32) indicates how the relaxation time  $\tau(E)$  is related to the collision term.

We distinguish the cases: 1) A classical plasma of low density  $n < \bar{n}$ , at any degree of nonideality  $\gamma/Z \lesssim 1$ , for which we expect the thermal energy of the electron to be much greater than  $\hbar\omega_s$ , i.e.,  $\hbar\omega_s \ll K T$ . In this case, only elastic scattering

of the electrons by the plasma waves is considered,  $E' = E$ . ii) A nonideal quantum plasma with  $\Gamma/Z \lesssim 1$  at high densities,  $n > \bar{n}$ , and for which in general, we can no longer neglect  $\hbar\omega_s$  compared to the electron energy  $E$ , i.e. for which  $\hbar\omega_s \gtrsim KT$ . It is recognized also by inspecting the relations of the frequencies to the wave vectors [Eqs. (12) and (18)] that the effects of the electron-electron interaction (electron-electron waves) and electron-ion interaction (electron-ion waves) on the relaxation time are to be studied as two distinct cases.

#### IV. CLASSICAL PLASMAS, $n < \tilde{n}$

The two frequency branches of the longitudinal oscillations are given by Eqs. (12) and (18) for the electron-electron interaction ( $s=e$ ) and electron-ion interaction ( $s=i$ ). The relaxation times  $\tau_{es}(E)$  ( $s=e,i$ ) are by Eqs. (21), (30), (32), (25), and (27)

$$\tau_{es}(E) = \frac{(\partial f_o / \partial E) \phi}{(\delta f / \delta t)_s}, \quad s=e,i, \quad (33)$$

where

$$\begin{aligned} \left( \frac{\delta f}{\delta t} \right)_s = & - \frac{\phi}{8\pi^2 m_s n_s K T} \int_{\mathbf{k}'} \frac{q^2 |U_s(q)|^2}{\omega_s(q)} \bar{N}_q^s \left\{ [f_o(\vec{k}') (1 - f_o(\vec{k}))] \delta(E' - E + \hbar\omega_s) \right. \\ & \left. + f_o(\vec{k}) [1 - f_o(\vec{k}')] \delta(E' - E - \hbar\omega_s) \right\} \left[ 1 - \frac{\phi'}{\phi} \right] d^3 \mathbf{k}'. \end{aligned} \quad (34)$$

For the evaluation of Eq. (34) we assume that  $\alpha) \hbar\omega_s \ll E$  (classical plasma),  $\beta) E' = E, |\vec{k}'| = |\vec{k}|$  (elastic scattering) and  $\gamma)$  isotropic scattering (no angular dependence before interaction). With these assumptions, the transformations in the Appendix A yield with Eqs. (33) and (34),

$$\tau_{es}(E) = \frac{8\sqrt{2}\pi m_s^{1/2} E^{3/2} n_s}{\hbar \int_0^{\hat{q}_s} \frac{q^5 |U_s(q)|^2 dq}{\omega_s(q) (\exp(\frac{\hbar\omega_s}{K T}) - 1)}}. \quad (35)$$

Before evaluating the integral in Eq.(35) for the two frequency branches under consideration, we first observe the behavior of  $|U_s(q)|^2$  in Eq.(6). In the  $q$ -domain,  $|U_s(q)|^2$  is bound between the limits

$$\frac{16\pi^2 n_s^2 e_s^2 \delta_s^4}{(1 + 4\pi^2)^2} \leq |U_s(q)|^2 \leq 16\pi^2 n_s^2 e_s^2 \delta_s^4, \quad s=e,i, \quad (36)$$

##### 1. Electron-Electron Wave Interaction

For the interaction of the electrons with the high frequency plasma oscillation of frequency  $\omega_s(q)$  [Eq. (12)], the relaxation time is evaluated by means of Eq.(35), and the plasmon distribution function  $\bar{N}_q^e$  of Eq.(1) as

$$\tau_{ee}(E) = \frac{\omega \sqrt{2} m^{3/2} E^{3/2}}{2\pi \hbar n e^4 I_e}, \quad (37)$$

$$I_e = \int_0^{\hat{q}_e} \frac{q^5 dq}{(\delta_e^{-2} + q^2)(1 + a^2 q^2)^{1/2} (\exp[\frac{\hbar \omega p}{KT}(1 + a^2 q^2)^{1/2}] - 1)},$$

$$= h(\tilde{q}_e) \int_0^{\hat{q}_e} \frac{q^5 dq}{(1 + a^2 q^2)^{1/2} (\exp[\frac{\hbar \omega p}{KT}(1 + a^2 q^2)^{1/2}] - 1)}, \quad 0 < \tilde{q}_e < \hat{q}_e, \quad (38)$$

where

$$h(\tilde{q}) = \frac{\delta_e^4}{(1 + \delta_e^2 \tilde{q}^2)^2} \approx (\delta_e / \pi)^4, \quad \hat{q}_e = 2\pi / \delta_e. \quad (39)$$

in accordance with the mean value theorem for integrals<sup>35)</sup> since  $U_s(q)$  is bounded [Eq. 36] in the interval  $(0, \hat{q}_e)$  and  $\tilde{q}_e$  is approximated by the mean value of the two limits of the integral. With a proper change of variables the integral in Eq. (38) is evaluated (Appendix B) as

$$I_e = \frac{4m\omega}{\hbar \kappa_e} R_e(\epsilon_p, a\hat{q}_e),$$

$$R_e(\epsilon_p, a\hat{q}_e) = 1 - \frac{2}{5} \hat{\epsilon}_e + 4 \sum_{v=1}^{\infty} \frac{B_{2v} \hat{\epsilon}_e^{2v}}{(2v+4)2v!} - \frac{1}{(a\hat{q}_e)^4} \left[ 1 - \frac{2}{5} \epsilon_p + 4 \sum_{v=1}^{\infty} \frac{B_{2v} \epsilon_p^{2v}}{(2v+4)2v!} \right]$$

$$- \frac{4}{(a\hat{q}_e)^2} \left[ 1 - \frac{\hat{\epsilon}_e}{3} + 2 \sum_{v=1}^{\infty} \frac{B_{2v} \hat{\epsilon}_e^{2v}}{(2v+2)2v!} \right]$$

$$+ \frac{4}{(a\hat{q}_e)^4} \left[ 1 - \frac{\epsilon_p}{3} + 2 \sum_{v=1}^{\infty} \frac{B_{2v} \epsilon_p^{2v}}{(2v+2)2v!} \right]$$

$$+ \frac{4}{(a\hat{q}_e)^4} \left[ \ln \left\{ \frac{\hat{\epsilon}_e}{\epsilon_p - 1} \right\} + \epsilon_p - \hat{\epsilon}_e \right] \quad (40)$$

$B_{2v}$  are Bernoulli numbers<sup>32)</sup>, and

$$\epsilon_p = \frac{\bar{\lambda}_e}{(1+2)d_D}, a\hat{q}_e = 2\pi\sqrt{\kappa_e} \frac{d_D}{\delta_e}, \quad \bar{\lambda}_e = \frac{\hbar}{(mKT)^{1/2}}, \quad \hat{\epsilon}_e = \epsilon_p (1 + a^2 \hat{q}_e^2)^{1/2} \quad (41)$$

where  $d_D$  is the Debye length. By Eqs. (37) and (38), the relaxation time due to the electron-electron wave interaction is

$$\tau_{ee}(E) = \frac{\sqrt{2} \pi^{1/2} \kappa_e E^{3/2}}{8 \pi n e^4 R_e(\epsilon_p, a \hat{q}_e)} \quad (42)$$

where  $R_e(\epsilon_p, a \hat{q}_e) \sim 1$  for  $n \ll \bar{n}$ .

## 2. Electron-Ion Wave Interaction

For the interaction of the electrons with the low frequency oscillations, of frequency  $\omega_i(q) = C_s q$  [Eq. (18)], the relaxation time  $\tau_{ei}(E)$  is evaluated by means of Eq. (35) and Eq. (18) with  $v(q) \sim 1$ , and  $|U_i(q)|$  from Eq. (6). By applying the mean value theorem for integrals as in Eq. (38), one obtains

$$\tau_{ei}(E) = \frac{C_s m^{1/2} \sqrt{2} M E^{3/2}}{2 \pi \hbar^2 n e^4 I_i} \quad (43)$$

where

$$I_i = \hbar(\hat{q}_i) \int_0^{\hat{q}_i} \frac{q^4 dq}{\left( \exp \left[ \frac{\hbar C_s q}{KT} \right] - 1 \right)}, \quad 0 \leq \tilde{q}_i \leq \hat{q}_i \quad (44)$$

$$\hat{q}_i = \frac{2\pi}{\delta_i}, \quad \hbar(\tilde{q}_i) \approx (\delta_i/\pi)^4 \quad (45)$$

The integral  $I_i$  is of Debye-type<sup>32)</sup>, which becomes with  $b = \frac{\hbar C_s}{KT}$  and  $bq = x$ ,

$$I_i = (\delta_i/\pi)^4 \frac{1}{b^5} \int_0^{b\hat{q}_i} \frac{x^4 dx}{e^x - 1} = (\delta_i/\pi)^4 \frac{\hat{q}_i^4}{4b} R_i(b\hat{q}_i) \quad (46)$$

$$R_i(b\hat{q}_i) = 1 - \frac{2b\hat{q}_i}{5} + 4 \sum_{v=1}^{\infty} \frac{B_{2v}(b\hat{q}_i)^{2v}}{(2v+2)2v!}, \quad b\hat{q}_i < 2\pi \quad (47)$$

Here again  $B_{2v}$  are Bernoulli numbers<sup>32)</sup>. The magnitudes of  $b$  and  $b\hat{q}_i$  follows from the relations,

$$b = \frac{\hbar C_s}{KT} = \sqrt{\kappa_i} \bar{\lambda}_i, \quad \bar{\lambda}_i = \frac{\hbar}{(MKT)^{1/2}}, \quad b\hat{q}_i = 2\pi\sqrt{\kappa_i} \lambda_i/\delta_i \quad (48)$$

Combination of Eqs. (42) and (44) yields, for the relaxation time due to electron-ion waves interactions:

$$\tau_{ei}(E) = \frac{\sqrt{2} m^{1/2} \kappa_i E^{3/2}}{8\pi n Z e^4 R_i(b\hat{q}_i)} , \quad (49)$$

where  $R_i(b\hat{q}_i) \sim 1$  for  $n \ll \tilde{n}$ .

### 3. Electrical Conductivity.

The electrical conductivity  $\sigma_c$  is related to the energy dependent relaxation times by:

$$\sigma_c = \frac{ne^2}{m} \langle \tau \rangle , \quad \frac{1}{\langle \tau \rangle} = \sum_s \frac{1}{\langle \tau_{es} \rangle} , \quad s=e,i , \quad (50)$$

where

$$\langle \tau_{es} \rangle = \int_0^\infty \tau_{es}(E) f(E) dE , \quad \int_0^\infty f(E) dE = 1 , \quad (51)$$

For classical plasmas,  $f(E)$  is the Maxwell distribution,  $f(E) = \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{(KT)^{3/2}} e^{-E/KT}$ , which gives

$$\langle \tau \rangle = \frac{m^{1/2} \kappa_e \kappa_i (kT)^{3/2}}{\sqrt{2\pi} n \pi e^4 [\kappa_i R_e(\epsilon_p, a\hat{q}_e) + Z \kappa_e R_i(b\hat{q}_i)]} , \quad (52)$$

From Eqs. (50) and (52), we obtain the conductivity for a nonideal classical plasma:

$$\sigma_c = \frac{\kappa_i \kappa_e (kT)^{3/2}}{\pi (2\pi m)^{1/2} e^2 L} , \quad (53)$$

where

$$L = \kappa_i R_e(\epsilon_p, a\hat{q}_e) + Z \kappa_e R_i(b\hat{q}_i) , \quad (54)$$

The above results will be discussed in section [VI].

#### 4. Thermal Conductivity of Nonideal Plasmas.

Charge and energy in a Coulomb system are transferred simultaneously during the motion of the electrons, no matter whether this motion is caused by an electrical field or a temperature gradient. In each case both an electrical and thermal current appear. The same methods used for the evaluation of the electrical conductivity may also be applied to the thermal conductivity. They are related by the Lorentz number  $(\pi^2/3)(K/e)^2$  through the Wiedemann-Franz law which assumes, however, that the collision processes are such that a common relaxation time exists for an electric and thermal field, which is always satisfied at high temperature i.e.  $\hbar\omega \ll KT$ , and hence

$$\frac{\lambda}{\sigma T} = \frac{\pi^2}{3} \left(\frac{K}{e}\right)^2, \quad (55)$$

and the thermal conductivity is simply by Eqs. (53) and (55)

$$\lambda = \frac{\kappa_e \kappa_i (\pi/2)^{1/2} K (KT)^{5/2}}{3e^4 m^{1/2} L}, \quad (56)$$

where  $L$  is given by Eq. (54).

## V. QUANTUM PLASMA, $n > \tilde{n}$ .

For plasma densities  $n < \tilde{n}$ , it was proper to ignore  $\hbar\omega_s$  compared with the energy  $E$  of the electrons which is of the order of magnitude of  $KT$ . At high pressures where  $n \gg \tilde{n}$ ,  $\hbar\omega_s$  can no longer be ignored compared to  $E$ .

Our approach to the quantum Boltzman equation follows the Kohler variational method<sup>33)</sup>, which is frequently used in connection with the resistivity of metals. Combining the collision term (30) with the electric field  $\vec{E}$  yields

$$\left(\frac{\delta f}{\delta t}\right)_s = \frac{1}{(2\pi)^3 KT} \int \bar{w}_s(\vec{k}', \vec{k}) [\phi(\vec{k}') - \phi(\vec{k})] d^3\vec{k}' = -v_x \frac{\partial f_o}{\partial E} eE, \quad (57)$$

Only  $v_x$  appears since  $\vec{E}$  is assumed in the  $\vec{x}$ -direction,  $\vec{E} = (E, 0, 0)$ .

According to Eqs. (32) and (55), (21) and (23), the interaction term  $(\delta f / \delta t)_s$  and the function  $\phi$  are

$$\left(\frac{\delta f}{\delta t}\right)_s = -v_x \frac{\partial f_o}{\partial E} eE, \quad (58)$$

$$\phi = -eE \frac{\partial f_o}{\partial E} v_x C(E). \quad (59)$$

In order to determine the function  $C(E)$ , we follow the idea of Bloch and expand  $C(E)$  in a power series of  $(E - \xi)$ .

$$C(E) = \sum_{\mu} C_{\mu} (E - \xi)^{\mu} = C_0 + C_1 (E - \xi) + \dots, \quad (60)$$

where this series is treated as a trial function in the Kohler variational method to determine the coefficients of the series  $C_{\mu}$ .

By Eq. (C-11) (Appendix C), the  $C_{\mu}$  are determined from the system of equations

$$\sum_{\nu} C_{\nu} D_{\mu\nu} = N_{\mu}, \quad (61)$$

with

$$D_{\mu\nu} = \frac{1}{(2\pi)^3 KT} \iint v_x (E - \xi)^{\mu} \bar{w}(\vec{k}', \vec{k}) [v'_x (E' - \xi)^{\nu} - v_x (E - \xi)^{\nu}] d^3\vec{k}' d^3\vec{k}, \quad (62)$$

and

$$N_{\mu} = - \int_0^{\infty} \frac{\partial f_0}{\partial E} (E - \xi)^{\mu} F(E) dE \quad (63)$$

$$F(E) = \int_{E=\text{const}} \int \frac{v_x^2 ds}{|dE/dk|}$$

where  $ds$  is a surface element.

Eq. (14) indicates that for the high frequency branch (electron oscillations) and  $n \gg \tilde{n}$ , the frequency  $\omega(q)$  depends on  $v_F$ , the Fermi speed (high pressure quantum plasma). Contrary to the classical case where  $n \ll \tilde{n}$ , in the quantum case the thermal energy of the electrons is small compared with the Fermi energy<sup>34)</sup>. Accordingly, the series expansion (59) is approximated by the first term  $C_0$ . This assumption is widely used in the transport theory of metals ("Bloch approximation") and gives good results especially for the electrical conductivity. Eqs. (32), and (56)-(61) give as relaxation time in the approximation  $C(E) \approx C_0$ ,

$$\tau_{es} = \frac{\int_0^{\infty} \frac{\partial f_0}{\partial E} F(E) dE}{\frac{1}{(2\pi)^3 K T} \iint v_x^2 \bar{W}(\vec{k}, \vec{k}) \left[1 - \frac{v_x}{v_x}\right] d^3 k d^3 k'} \approx \frac{N_0}{D_{00}} \quad (64)$$

where  $N_0$  and  $D_{00}$  are to be evaluated. The integral  $N_0$  in the numerator of Eq. (62) is of Fermi-type and since the contribution to the integral arises in the vicinity of  $E = \xi$ , the limits have been extended to 0 and  $\infty$ . Following the approximation made by Haug<sup>22)</sup>, we have

$$N_0 = F(E) \Big|_{E=\xi} + O((KT)^2) \quad (65)$$

where  $F(E)$  is given in Eq. (61), with  $v_x^2 = \hbar^2 k^2 (\cos \alpha / m)^2$ ,  $ds = k^2 \sin \alpha d\alpha d\phi$  and  $dE/dk = \hbar^2 k / m$  (the angles  $\alpha$ , and  $\phi$  are defined in Appendix A.). Accordingly,

$$F(E) = \frac{k^3}{4\pi} \int_{\alpha=0}^{\pi} \int_{\phi=0}^{2\pi} \cos^2 \alpha \sin \alpha d\alpha d\phi = \frac{4\pi}{3m} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{3/2} \quad (66)$$

i.e.

$$N_0 = \frac{4\pi}{3m} \left( \frac{2m}{\hbar^2} \right)^{3/2} (\xi_0)^{3/2}, \quad \xi_0 = (3\pi^2 n)^{2/3} \frac{\hbar^2}{2m}, \quad (67)$$

The denominator  $D_{00}$  of Eq. (62) contains  $\bar{W}(\vec{k}, \vec{k}')$  defined in Eq. (29), which contains the transition probability  $P(\vec{k}', \vec{k})$ . This latter function has different forms for the absorption [Eq. (25)] and for the emission [Eq. (27)] of a plasmon. First consider the case of the absorption of a plasmon with  $E' = E + \hbar\omega_s$ , and  $D_{00} \equiv D_{00}^+$ .

Using Eq. (25) and (29) in the denominator of Eq. (62) we get after integrating over  $E$

$$D_{00}^+ = \frac{\pi}{(2\pi)^3 K T n_s m_s} \iiint \frac{v_x^2 q^2 |U_s(q)|^2 \bar{N}_q^s}{\omega_s(q)} f_0(E) [1 - f_0(E + \hbar\omega_s)] \cdot \left[ 1 - \frac{v'_x}{v_x} \right] \frac{ds'}{\hbar v'} \frac{ds}{\hbar v} dE, \quad (68)$$

where  $E' = E + \hbar\omega_s$ ,  $ds'$  and  $ds$  are elements of the energy surface. With  $\alpha$  = angle between  $\vec{k}$  and the  $\vec{x}$ -direction,  $\gamma$  = angle between  $\vec{k}'$  and the  $x$ -direction,  $\theta$  = angle between  $\vec{k}$  and  $\vec{k}'$ , and  $\phi, \psi$  = azimuthal angles around  $\vec{k}$  and  $\vec{k}'$  respectively,  $ds$  and  $ds'$  are expressed as

$$ds = k^2 \sin\alpha \, d\alpha \, d\phi = \frac{E}{\hbar^2} 2m \sin\alpha \, d\alpha \, d\phi, \quad (69)$$

$$ds' = k'^2 \sin\gamma \, d\gamma \, d\psi = \frac{k'}{k} q \, dq \, d\psi, \quad (70)$$

where

$$\begin{aligned} \frac{v'_x}{v_x} &= \frac{v' \cos\alpha'}{v \cos\alpha} = \frac{k'}{k} (\cos\gamma + \tan\alpha \sin\gamma \cos\psi), \\ \frac{k'}{k} \cos\gamma &= 1 + \frac{\hbar\omega_s - \hbar^2 q^2 / 2m}{2E}, \\ \frac{v_x^2}{\hbar^2 v' v} &= \frac{k}{k'} \frac{\cos^2\alpha}{\hbar^2}, \end{aligned} \quad (71)$$

After evaluating the angular integrals and expressing  $\bar{N}_q^s$ ,  $|U(q)|^2$  by their respective expressions in Eqs.(1) and (6), we obtain

$$D_{oo}^+ = \frac{32\pi^2 m_s n_s e^2 e^2}{3\hbar^4 m_s KT} \int_{-\infty}^{+\infty} \int_0^{\hat{q}_s} \frac{q^3 f_o(E) [1 - f_o(E + \hbar\omega_s)]}{(\delta_s^{-2} + q^2)^2 \omega_s(q) [\exp(\frac{\hbar\omega_s}{KT}) - 1]} \cdot \left[ \frac{\hbar^2 q^2}{4mE} - \frac{\hbar\omega_s}{2E} \right] E dE dq. \quad (72)$$

The double integral of Eq. (72) is evaluated using the mean value theorem<sup>35)</sup> as

$$h(\tilde{q}_s) \int_{-\infty}^{+\infty} \int_0^{\hat{q}_s} \frac{q^3 f_o(E) [1 - f_o(E + \hbar\omega_s)]}{\omega_s(q) [\exp(\frac{\hbar\omega_s}{KT}) - 1]} \left[ \frac{\hbar^2 q^2}{4mE} - \frac{\hbar\omega_s}{2E} \right] E dE dq, \quad 0 < \tilde{q}_s < \hat{q}_s \quad (73)$$

where

$$h(\tilde{q}_s) = \frac{\delta_s^4}{(1 + \delta_s^2 \tilde{q}_s^2)^2} \approx (\delta_s / \pi)^4. \quad (74)$$

In Eq.(70),  $\omega_s(q)$  ( $s=e$ ) is the dispersion of the longitudinal waves of the degenerate ( $n \gg \bar{n}$ ) electrons and  $f(E)$  is the Fermi distribution. With the change of variables,

$$\eta = \frac{(E - \xi)}{KT}, \quad \epsilon = \frac{\hbar\omega_s}{KT}, \quad \omega_s = \frac{KT}{\hbar} \epsilon, \quad \epsilon_p = \frac{\hbar\omega_p}{KT}, \quad (75)$$

which gives

$$q^3 dq = \frac{1}{(a\epsilon_p)^4} \epsilon (\epsilon^2 - \epsilon_p^2) d\epsilon,$$

$$E = \eta KT + \xi, \quad dE = KT d\eta,$$

$$\frac{\hbar\omega_s}{2E} = \frac{\epsilon KT}{2(\eta KT + \xi)},$$

$$\frac{\hbar^2 q^2}{4mE} = \frac{\hbar^2 (\epsilon^2 - \epsilon_p^2)}{4m a^2 \epsilon_p^2 (\eta KT + \xi)},$$

$$f(E) [1 - f(E + \hbar\omega_s)] = \frac{1}{(e^\eta + 1)(1 + e^{-\eta - \epsilon})} = \frac{1}{(1 - e^{-\epsilon})} \left[ \frac{1}{e^\eta + 1} - \frac{1}{e^{\eta + \epsilon} + 1} \right], \quad (76)$$

Eq.(72) becomes

$$D_{oo}^+ = \frac{32\pi e^4 \delta^4 e}{3\pi^2 \hbar^3 KT (a\epsilon_p)^4} \int_{\epsilon_p}^{\hat{\epsilon}_e} (\epsilon^2 - \epsilon_p^2) \left[ \frac{\hbar^2 (\epsilon^2 - \epsilon_p^2)}{4m a^2 \epsilon_p^2} - \frac{\epsilon KT}{2} \right] \frac{H(\epsilon)}{1 - e^{-\epsilon}} d\epsilon$$

$$H(\epsilon) = \int_{-\infty}^{+\infty} \frac{d\eta}{(e^\eta + 1)(e^{-\eta - \epsilon} + 1)} \quad , \quad (77)$$

where

$$\hat{\epsilon}_e = \epsilon_p (1 + a^2 \hat{q}_e^2)^{1/2} \quad . \quad (78)$$

The integral over  $\eta$  in the  $H$ -integral is easily carried out and is equal to  $\epsilon/(1 - e^{-\epsilon})$ , while the integral over  $\epsilon$  is developed in a series of integrals of Einstein type<sup>32)</sup>,

$$J_v = \int_{\epsilon_p}^{\hat{\epsilon}_e} \frac{x^v e^x dx}{(e^x - 1)^2} \quad . \quad (79)$$

Eq. (79) is evaluated by expanding the denominator in a series of  $e^{-x}$

$$J_v = \sum_{n=1}^{\infty} \int_{\epsilon_p}^{\hat{\epsilon}_e} n x^v e^{-nx} dx = \sum_{n=1}^{\infty} \left\{ e^{-n\epsilon_p} \left[ \epsilon_p^v + \frac{v\epsilon_p^{v-1}}{n} + \dots + \frac{v!}{n^v} \right] - e^{-n\hat{\epsilon}_e} \left[ \epsilon_e^v + \frac{v\epsilon_e^{v-1}}{n} + \dots + \frac{v!}{n^v} \right] \right\} \quad . \quad (80)$$

Substitution of (80) into (77) shows that

$$D_{oo}^+ = \sum_n a_n J_n \quad , \quad (81)$$

where the  $a_n$  are to be defined shortly, and the sum over  $n$  has only a few terms. For the emission transition,  $\vec{k}' = \vec{k} - \vec{q}$ ,  $E' = E - \hbar\omega_s$ ,  $D_{oo}^-$  is obtained from  $D_{oo}^+$  by replacing  $\vec{q}$  by  $-\vec{q}$  and  $\omega_s$  by  $-\omega_s$ , that is  $\epsilon$  by  $-\epsilon$ , and by adding a factor  $(-1)^{n+1}$  to the numerator. Accordingly

$$D_{oo}^- = \sum_n (-1)^{n+1} a_n J_n \quad . \quad (82)$$

The total denominator  $D_{oo} = D_{oo}^- + D_{oo}^+$  of Eq. (62) is by Eqs. (81) and (82)

$$D_{oo} = \sum_n (1 + (-1)^{n+1}) a_n J_n \quad (83)$$

The terms with an even power of  $\epsilon$  in the integrand of Eq.(77) vanish according to (83). Only terms of odd power remain whose coefficients are the same as those of  $D_{oo}^+$  apart from a factor 2. Hence,

$$D_{oo} = \frac{2^4 n e^4 \delta_e^4}{3\pi^2 m \hbar K T (a \epsilon_p)^6} \int_{\epsilon_p}^{\hat{\epsilon}_e} \frac{(\epsilon^2 - \epsilon_p^2)^2 \epsilon e^\epsilon d\epsilon}{(e^\epsilon - 1)^2} \quad (84)$$

In the range of pressures and temperatures in which this equation will be applied, the variables of interest will be  $\epsilon_p$  given by Eq(75) and  $\hat{a}q_e$  both of which can be expressed in terms of nondimensional parameters, with  $\hat{q}_e = 2\pi/\delta_e$ ,

$$\hat{a}q_e = (9/5)^{1/2} (2\pi)^{2/3} (n/\tilde{n})^{1/3} (\gamma/Z)^{-1/2}, \quad (85)$$

$$\epsilon_p = 2\pi^2^{5/6} (n/\tilde{n})^{1/3} (\gamma/Z)^{1/2}. \quad (86)$$

The integral in Eq. (83) is a sum of integrals  $J_v$ ,  $v = 5, 3, 1$ , defined in Eq.(81), with different coefficients  $a_n$ , namely  $a_5 = 1$ ,  $a_3 = -2\epsilon_p^2$ ,  $a_1 = \epsilon_p^4$ .

Hence

$$D_{oo} = \beta [J_5 - 2\epsilon_p^2 J_3 + \epsilon_p^4 J_1] = \beta F(\epsilon_p, \hat{a}q_e), \quad (87)$$

where

$$\beta = \frac{2^4 n e^4 \delta_e^4}{3\pi^2 m \hbar K T (a \epsilon_p)^6} \quad (88)$$

The relaxation time for electron-electron wave interaction in a degenerate quantum plasma (Eq.(62)) is obtained from Eq.(65), (87) and (88) in the form

$$\tau_{ee} = \frac{3\pi^5 \hbar K T (a \epsilon_p)^6}{e^4 \delta_e^4 F(\epsilon_p, \hat{a}q_e)} \quad (89)$$

At high pressures for which  $\epsilon_p > 1$ , we can easily terminate the series of the function  $F(\epsilon_p, \hat{a}q_e)$ ,

$$F(\epsilon_p, a\hat{q}_e) \approx 8\epsilon_p^3 \exp(-\epsilon_p) \left[1 + \frac{6}{\epsilon_p}\right] + O(e^{-\epsilon_p}/\epsilon_p^2), \quad \epsilon_p > 1. \quad (90)$$

$$\text{Letting } (1 + \frac{6}{\epsilon_p}) = \alpha, \text{ and } F(\epsilon_p, a\hat{q}_e) \approx 8\alpha \epsilon_p^3 e^{-\epsilon_p}, \quad (91)$$

$$\tau_{ee} = \frac{3\pi^5 \hbar^3 K T a^6 \epsilon_p^3}{8e^4 \alpha \delta_e^4 e^{-\epsilon_p}} \quad , \quad \epsilon_p > 1, \text{ for } n > \tilde{n} \text{ and } \Gamma/Z \gg 1, \quad (92)$$

where  $a$ ,  $\epsilon_p$ ,  $\delta_e$  are given by Eqs. (14), (43) and (10) respectively. For the quantum plasma, the ions are still classical, and a relaxation time resulting from the electron-ion sound wave interaction can be derived from Eq. (43) with  $E = \xi_0$  (Fermi energy) since  $\hbar\omega_i \ll E$  for all densities under consideration. Thus

$$\tau_{ei} = \frac{3\pi \kappa_i \hbar^3}{2^4 m Z e^4 R_i(b\hat{q}_i)} \quad (93)$$

For quantum plasmas  $b\hat{q}_i$  can be quite large, so that the series in Eq. (47) does not converge for  $b\hat{q}_i > 2\pi$ . Hence in the present case where  $b\hat{q}_i \geq 2\pi$  ( $b\hat{q}_i = x$ ),

$$R_i(x) = 1 - \frac{2x}{5} + 4 \sum_{v=1}^{\infty} \frac{B_{2v} x^{2v}}{(2v+2)2v!} \quad , \quad x < 2\pi \quad , \quad (94)$$

$$R_i(x) = \frac{96\zeta(5)}{x^4} - 4 \sum_{v=1}^{\infty} e^{-vx} \left[ \frac{1}{v} + \frac{4}{v^2 x} + \frac{12}{v^3 x^2} + \frac{24}{v^4 x^3} + \frac{24}{v^5 x^4} \right] \quad , \quad x \geq 2\pi \quad , \quad (95)$$

where  $\zeta(5) = \sum_{k=1}^{\infty} k^{-5}$  is the Riemann Zeta-function<sup>32)</sup>.

The effective relaxation time due to both electron-electron wave and electron-ion wave interactions in a nonideal quantum plasma is

$$\tau_{eff} = \frac{\tau_{ei} \tau_{ee}}{\tau_{ee} + \tau_{ei}} = \frac{\tau_{ei}}{1+G(T)} \quad , \quad (96)$$

where the relaxation time ratio is

$$G(T) = \frac{\tau_{ei}}{\tau_{ee}} \quad (97)$$

For  $\xi_0 \rightarrow E \sim kT$  and  $\hbar\omega_e \ll kT$ , Eq. (92) and (93), averaged over a Maxwell distribution give the classical electrical conductivity derived in Eq. (53). On the other hand, if we

assume  $\hbar\omega_e > kT$  and complete degeneracy of the electrons, the electrical conductivity (50) of the quantum plasma with the relaxation time (96) is

$$\sigma_Q = \frac{3\pi n k_1 \hbar^3}{2^4 m^2 Z e^2 R_i (\hat{b} q_i) (1+G(T))}, \quad (98)$$

where for all cases of interest, i.e.,  $10^{21} \text{ cm}^{-3} \leq n \leq 10^{24} \text{ cm}^{-3}$  and  $T = 10^3 - 10^4 \text{ K}$ ,

$G(T) \ll 1$ . But for higher densities and lower temperatures  $G(T)$  can make a significant contribution. The electrical conductivity  $\sigma_Q$  is a linear function of the density  $n$  but less sensitive to the temperature  $T$ . Eq.(98) agrees with the expression for electrical conductivity of a low temperature metal<sup>25)</sup>. Further discussions and applications follow in section VI.

## VI. APPLICATIONS

### 1. Classical nonideal plasmas.

In order to apply the formulas derived above, we propose a study of the electrical conductivity as a function of the relevant parameters involved. For applications of the theory to strongly, intermediate and weakly nonideal plasmas, it should be noted that the dimensionless  $\gamma/Z$ ,  $\epsilon_p = \hbar\omega_p/KT$ ,  $a\hat{q}_e$ ,  $b\hat{q}_i$  and  $n/\tilde{n}$  occurring in Eqs. (53) and (54) for the electrical conductivity of classical nonideal plasmas can not be varied independently. Since  $\gamma/Z$  increase with increasing  $n$  and decreases with increasing  $T$ ,  $\epsilon_p$  varies over a large  $n$ - $T$  region and hence, so does  $\hat{\epsilon}_e$ , similar to  $\gamma/Z$ . Numerically,  $\gamma/Z = 1.67 \times 10^{-3} n^{1/3} T^{-1}$ ,  $\epsilon_p = 4.328 \times 10^{-7} n^{1/2} T^{-1}$ ,  $n/\tilde{n} = 2.07 \times 10^{-16} n T^{-3/2}$ , (99)

where for  $\gamma/Z \geq 1$ ,

$$a\hat{q}_e = (4\pi/3)^{1/3} (\pi\kappa_e)^{1/2} (\gamma/Z)^{-1/2}, \quad (100)$$

$\epsilon_p$  is given by Eq. (86).

For  $\gamma/Z < 1$ ,  $a\hat{q}_e \gg 1$  and hence from Eq. (40)  $\hat{\epsilon}_e \approx \epsilon_p a\hat{q}_e$ , i.e.,

$$\hat{\epsilon}_e = \pi^{11/6} (32\kappa_e)^{1/2} (n/\tilde{n})^{1/3} \quad (101)$$

and for  $\gamma/Z \geq 1$ ,  $a\hat{q}_e \leq 1$ , Eq. (40) reads,

$$\epsilon_e = 2\pi(32)^{1/6} (\gamma/Z)^{1/2} [1 + (4\pi/3)^{2/3} \pi\kappa_e (\gamma/Z)^{-1}]^{1/2} (n/\tilde{n})^{1/3}, \quad (102)$$

In accordance with Eqs. (102) and (103), it is clearly seen that  $R_e(\epsilon_p, a\hat{q}_e) \approx 1$  for all  $\gamma/Z \geq 1$ , as long as  $n < \tilde{n}$ .

So far the parameters studied are related to the electron-electron interaction.

For the electron-ion contribution only one characteristic parameter  $b\hat{q}_i$  occurs in  $R_i(b\hat{q}_i)$  of Eq. (45). For this case of relatively low density plasma,  $n < \tilde{n}$  and  $\gamma/Z \leq 1$ ,  $\delta_i$  given by Eq. (8); the expression of  $b\hat{q}_i$  ( $\hat{q}_i = 2\pi/\delta_i$ ) as a function of the relevant dimensionless parameters is

$$b\hat{q}_i = (2\pi)^{3/2} (8\pi/3Z)^{1/3} \kappa_i^{1/2} (m/M)^{1/2} [1 + Z_0 (\gamma/Z)^{-1/2}]^{-1} (n/\tilde{n})^{1/3} \quad (103)$$

$$Z_0 = (4\pi/3Z)^{1/3} [4\pi(1+Z)]^{-1/2}, \quad (104)$$

and numerically,

$$b\hat{q}_i = 4.129 \times 10^1 (m/M)^{1/2} Z^{-1/3} [1 + Z_0 (\gamma/Z)^{-1/2}]^{-1} (n/\tilde{n})^{1/3} \quad (105)$$

Hence  $b\hat{q}_i \ll 1$  for  $n < \tilde{n}$  and  $R_i(b\hat{q}_i) \approx 1$  according to Eq. (45). In view of this order of magnitude of  $Re(\epsilon_p, a\hat{q}_e)$  and  $R_i(b\hat{q}_i)$  as  $n/\tilde{n} \rightarrow 0$ , the electrical conductivity of the classical nonideal plasma [Eq. (53)] is a very weakly varying function of the shielding parameters  $\delta_s$  ( $s=e, i$ ) defined in Eqs. (8), (10), and (11). The weak dependence is attributed to the many-body interaction character of the present theory, since the electron is interacting with the plasma as a whole rather than with individual particles.

The electrical, conductivity formula presently derived for classical nonideal plasmas is only weakly dependent on the density of the electrons  $n$ , and goes as  $\sim T^{3/2}$ . Hence, in this respect it does not only agree with the usual kinetic theory results  $\frac{1-11}{}$  (ideal plasmas) as  $n$  gets very small compared to  $\tilde{n}$  or  $\gamma/Z \ll 1$ , but also with the recent theories for nonideal plasmas as well  $\frac{1-2, 17}{}$ .

The nonideal effects of the plasma on the electrical conductivity are then governed by the Coulomb interaction alone through the nondimensional Coulomb conductivity  $\sigma^*(\gamma/Z)$  which we define by

$$\sigma_c = \sigma_c^* (\gamma/Z) \frac{(kT)^{3/2}}{m^{1/2} e^2}, \quad (106)$$

and with Eq. (53) it is shown that,

$$\sigma_c^* (\gamma/Z) = \frac{\kappa_e \kappa_i / \sqrt{2\pi}}{L(\gamma/Z, n/\tilde{n})} \quad (107)$$

$L(\gamma/Z, n/\tilde{n})$  is defined in Appendix D as

$$L(\gamma/Z, n/\tilde{n}) = \kappa_i R_e(\gamma/Z, n/\tilde{n}) + Z \kappa_e R_i(\gamma/Z, n/\tilde{n}) \quad (108)$$

where for  $n < \tilde{n}$

$$R_e(\gamma/Z, n/\tilde{n}) = 1 + \mu_1 (n/\tilde{n})^{1/3} [1 + a_0 (\gamma/Z)^{-1}]^{1/2} (\gamma/Z)^{1/2} + a_1 (\gamma/Z) + \dots, \quad (109)$$

and

$$R_i(\gamma/Z, n/\tilde{n}) = 1 - b_0 [1 + Z_0 (\gamma/Z)^{-1/2}]^{-1} (n/\tilde{n})^{1/3} + b_1 [1 + Z_0 (\gamma/Z)^{-1/2}]^{-2} (n/\tilde{n})^{2/3} + \dots, \quad (110)$$

The constants  $\mu_1, b_0$  etc., are given in Appendix D. Comparison with the existing

results such as ideal plasma  $\frac{1-11}{\sigma^*(\gamma/Z)}$  would be  $\sigma_s^*(\gamma/Z) = [3(2/\pi)^{1/2}/4] \ln[\alpha_0(\gamma/Z)^{-3/2}]$  with  $\alpha_0 = 3/Z(4\pi(1+Z))^{1/2}$  and is valid only for  $\gamma/Z \ll 1$ . Similar nondimensional Coulomb conductivities can be expressed from the formulas which are of special applicability to only weakly and intermediate nonideal classical plasmas, i.e.,  $\frac{\gamma}{Z} < 1$ . From the first case (Ebeling et al<sup>1</sup>),  $\sigma^*(\gamma/Z)$  can be shown to be:

$$\sigma_E^*(\gamma/Z) = [1 - (1/2)Z\alpha_0](\gamma/Z)^{3/2} / f \ln[\alpha_0(\gamma/Z)^{-3/2}], \quad (111)$$

where  $f = 1.73$ . And from the second case (Wilhelm<sup>17</sup>),  $\sigma^*(\gamma/Z)$  is:

$$\sigma_c^*(\gamma/Z) = \frac{3/(8\pi)^{1/2}Z}{\ln\left[\frac{8KT}{\hbar^2/m\delta_\lambda^2}\right]}, \quad \delta_\lambda = \left(\frac{4\pi n}{3Z}\right)^{-1/3} \left\{ 1 + \left(\frac{4\pi}{3Z}\right)^{1/3} [4\pi(1+Z)]^{-1/2} (\gamma/Z)^{-1/2} \right\}, \quad (112)$$

where  $\delta_\lambda = d_D$  for  $\gamma/Z \ll 1$  and  $\delta_\lambda = (4\pi n/3Z)^{-1/3}$  for  $\gamma/Z \gg 1$ . Table I compares these different nondimensional Coulomb conductivities over a wide range of densities expressed as a ratio  $n/\bar{n}$  at typical plasma temperature  $T = 10^4$  K with  $Z=1$ . At high densities both  $\sigma_s^*$  and  $\sigma_E^*$  show a sudden jump at  $n > 10^{19} \text{ cm}^{-3}$ , and would give a negative value at  $n > 10^{20} \text{ cm}^{-3}$  and thereby their applicability comes to an end at these and higher densities.  $\sigma_w^*(\gamma/Z)$  on the other hand, shows significant nonideal effects of the plasma, which can be attributed to the argument of the logarithm which is  $\propto n^{-2/3}$  for  $\gamma/Z \gg 1$  and behaves like the ideal Coulomb logarithm argument for  $\gamma/Z \ll 1$  which is  $\propto n^{-1/2}$ . This behavior is observed in Fig. 1 where we draw  $\sigma_w^*(\gamma/Z)$  as a function of  $\gamma/Z$  along with the other Coulomb conductivities.  $\sigma_E^*(\gamma/Z)$  shows evidence of its limitation to only weakly nonideal plasmas as it behaves (besides a very small difference in magnitude) identically with  $\sigma_s^*(\gamma/Z)$  of the ideal plasma.  $\sigma_c^*(\gamma/Z)$  of the present theory, on the other hand, shows much more evidence of the effects of the nonideality of the electrical conductivity. It shows an important difference in magnitude for  $\frac{\gamma}{Z} > 1$  and yet converges to  $\sigma_s^*(\gamma/Z)$  of the ideal plasma as  $\gamma/Z \rightarrow 0$ .  $\sigma_w^*(\gamma/Z)$  does not show this later behavior as  $\gamma/Z \rightarrow 0$  which is due to the quantum mechanical scattering involved. Curves similar to 3 and 4 of Fig. 1 have been reported in experiments<sup>29,30</sup>.

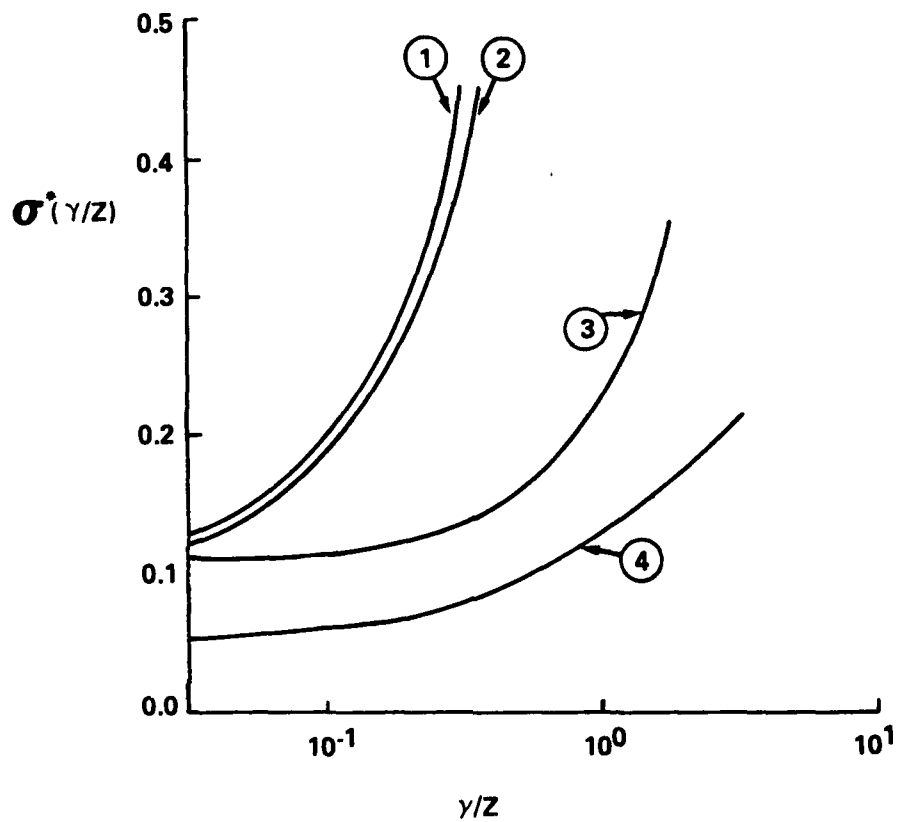


Fig. 1. Dimensionless Coulomb Conductivity Versus Interaction Parameter  $\gamma/Z$  of Different Theories at  $T=10^4$  °K.  
 1:ideal plasma, 2:[1] 3:present theory, 4:[17].

**TABLE I:** The Nondimensional Coulomb Conductivity  $\sigma^*(\gamma/Z)$

at  $T=10^4$  °K,  $Z=1$ .

$n/\bar{n}$	$\sigma_c^*$ present theory	$\sigma_s^*$ ideal	$\sigma_w^*$ [17]	$\sigma_E^*$ [1]
$8.40 \times 10^{-7}$	0.109	0.130	0.045	0.129
$8.40 \times 10^{-6}$	0.102	0.175	0.053	0.173
$8.40 \times 10^{-5}$	0.120	0.268	0.064	0.261
$8.40 \times 10^{-4}$	0.139	0.571	0.080	0.530
$8.40 \times 10^{-3}$	0.191	-	0.105	-

Fig. 2 shows the electrical conductivity of classical nonideal plasma presently derived and given by Eq. (53), versus the variable interaction parameter  $\gamma/Z$  and constant density  $n$ . It is seen that the value of  $\sigma_c$  is slashed by several order of magnitude as  $\gamma/Z$  goes from  $\sim 10^{-3}$  to  $\gamma/Z \geq 10$ , through variation of the temperature  $T$  at a fixed density  $n$ . It should be noted however that a slower increase is expected in the electrical conductivity  $\sigma_c$  with increasing  $\gamma/Z$  by varying the density  $n$  at fixed  $T$ . This later behavior is due to the dependence of the Coulomb conductivity  $\sigma_c^*$  on  $\gamma/Z$  and  $n/\bar{n}$  through  $L(\gamma/Z, n/\bar{n})$ . Such an increase would not have existed if  $L(\gamma/Z, n/\bar{n})$  were a constant, since  $\gamma/Z \sim T^{-1} n^{1/3}$  and  $\sigma_c \sim \sigma_c^*/(\gamma/Z)^{3/2}$ . A numerical illustration of this point and of the order of magnitude of  $\sigma_c$  is shown in Table II for a typical Temperature  $T = 10^4$  °K at different densities for a H-plasma. The relation between the cgs units and the practical units is  $9 \times 10^{11} \text{ mho-cm}^{-1} = 1 \text{ sec}^{-1}$ .

**TABLE II:** The Electrical Conductivity  $\sigma_c$  at  $T=10^4$  °K and  $Z=1$ .

$n \text{ cm}^{-3}$	$\gamma/Z$	$\sigma_c \text{ mho-cm}^{-1}$
$10^{17}$	0.077	$2.930 \times 10^1$
$10^{19}$	0.36	$3.630 \times 10^1$
$10^{21}$	1.67	$8.590 \times 10^1$

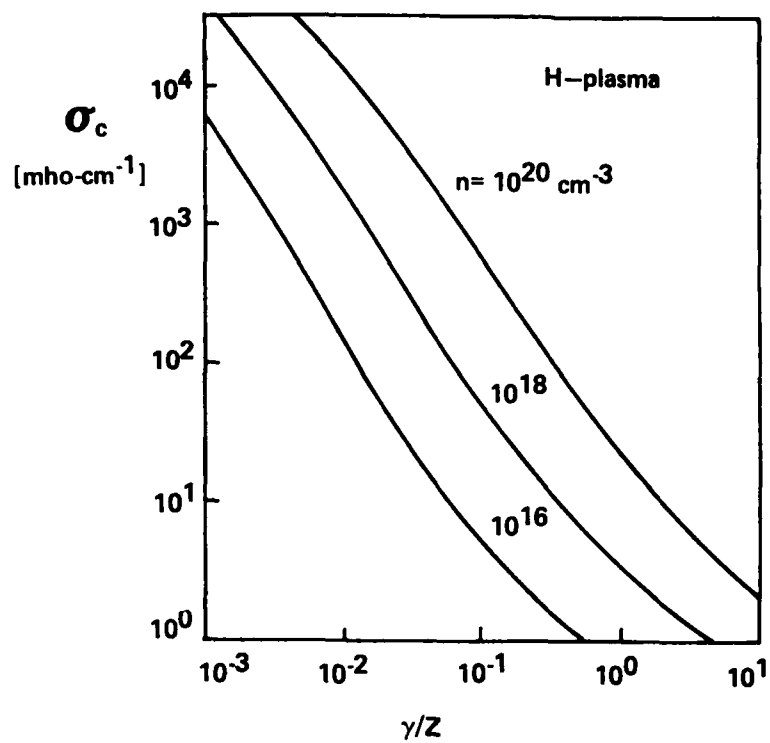


Fig. 2. Electrical Conductivity [mho-cm<sup>-1</sup>] of Classical Nonideal Plasma Versus the Interaction Parameter  $\gamma/Z$  for Constant Density  $n$ .

## 2. Comparison with Experiments

The alkali plasmas are being the more commonly used in the measurements, due to their lower ionization potential and the material characteristics they offer as wires, powders and vapors. Although, a bulk of experimental results has been accumulated over the years on the electrical conductivity<sup>5,12,15,29</sup>, the measurements reported are mostly those of weakly nonideal plasmas, i.e.  $0 < \gamma/Z < 1$ . Here we present in Fig. 3 the Coulomb conductivity given by Eq.(107) corresponding to a classical, weakly nonideal Lithium plasma. For Lithium (first ionization potential  $I_1 = 5.39$  ev) conductivity data are reported in Refs. 36,37. By letting  $KT[1+(1/Z)] = p/n$ , where  $p$  is the plasma pressure we have

$$n = \left[ \frac{\gamma p}{(Z+1)e^2} \right]^{3/4} \quad (113)$$

and

$$n/\bar{n} = \alpha_1 p^{3/8} (\gamma/Z)^{15/8}, \quad \alpha_1 = (\pi^{3/2} (2\pi m)^{3/2} e^{15/4} [1+(1/Z)]^{3/8} \quad (114)$$

Thus Eq.(107) becomes

$$\sigma_c^* (\gamma/Z, p) = \frac{\kappa_e \kappa_i / (2\pi)^{1/2}}{\pi L(\gamma/Z, p)}, \quad (115)$$

where

$$\begin{aligned} L(\gamma/Z, p) = (5/3) \left\{ 2 + \mu_1 \alpha_1^{1/3} [1+a_0(\gamma/Z)^{-1}]^{1/2} (\gamma/Z)^{19/8} p^{1/8} + a_1 (\gamma/Z)^1 \right. \\ \left. + \mu_2 \alpha_1^{2/3} [1+a_0(\gamma/Z)^{-1}]^1 (\gamma/Z)^{13/8} p^{1/4} - b_0 \alpha_1^{1/3} [1+Z_0(\gamma/Z)^{-1/2}]^{-1} \cdot \right. \\ \left. \cdot (\gamma/Z)^{5/8} p^{1/8} + b_1 \alpha_1^{2/3} [1+Z_0(\gamma/Z)^{-1/2}]^{-2} (\gamma/Z)^{5/4} p^{1/4} + \dots \right\}, \end{aligned} \quad (116)$$

with  $\alpha_1$  defined in Eq (114),  $a_0$ ,  $a_1$  etc... are given in Appendix D, and  $\kappa_e = \kappa_i = 5/3$ . In Fig. 3  $\sigma_c^*$  is shown for two different pressures 500 and 125 atmospheres at  $Z = 1$ . The theoretical curves are isobars in the low interaction parameter range which agree with the experimental results for a Lithium plasma. Along with  $\sigma_c^*$ ,  $\sigma_w^*$  given by Eq. (112) is depicted and its agreement with the

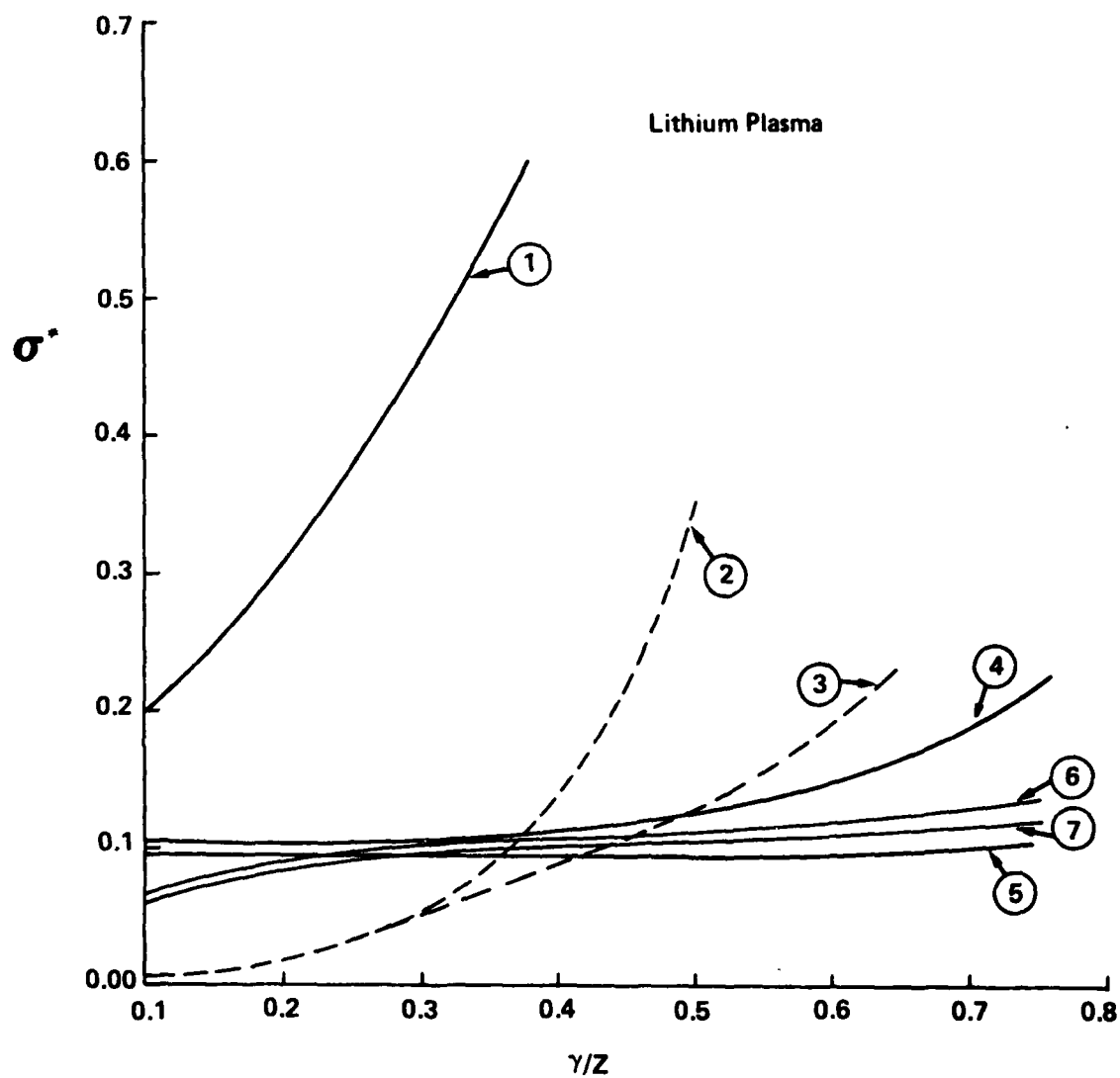


Fig. 3. Electrical Conductivity of Nonideal Plasma.  
 1) Spitzer's Formula; Experiment [36]: 2) 125, 3) 500 atm  
 Present Theory: 4) 125, 5) 500 atm  
 [17]: (6) 500, 7) 125 atm

experimental results is also good. One can observe, however, that  $\sigma_w^*$  shows a slight increase of the Coulomb conductivity with increasing the pressure  $p$  over the entire range  $0 \leq \gamma/Z < 1$ . The pressure dependence of  $\sigma_w^*$  is given by:

$$\Lambda_w = (8m/\hbar^2) (4\pi/3Z)^{-2/3} \frac{(Z+1)^{1/4}}{Z^{1/4}} (Ze^2)^{5/4} \frac{\gamma^{-5/4}}{p^{1/4}} [1+c\gamma^{-1/2}]^2, \quad (117)$$

with  $c = (4\pi/3Z)^{1/3} [Z/4\pi(1+Z)]^{1/2}$ , and hence

$$\sigma_w^* = (3/4Z)(2/\pi)^{1/2} / \ln \Lambda_w. \quad (118)$$

It should be noted that while  $\sigma_c^*$ ,  $\sigma_w^*$  and the experimental results of Fig. 3 are in a close agreement in the range  $0.3 \leq \gamma/Z \leq 0.5$ , the Coulomb conductivity  $\sigma_s^*$  of the ideal plasma is clearly inadequate over the entire range of the non-ideal regime.

### 3. Nonideal Quantum Plasmas

As it was shown in section V, in connection with the relaxation time ratio  $G(T)$  given by Eq. (97), the electron-electron wave interaction contributes little to alter the relaxation time of the electron in high density plasmas in the range  $n=10^{20}$ – $10^{24}$  cm<sup>-3</sup>, i.e.,  $G(T) \ll 1$ . The electrical conductivity is mainly due to electron-ion waves interaction. In order to apply the conductivity formula (98) to nonideal quantum plasmas, we express the dimensionless formula  $\sigma_Q/\omega_p$  as a function of the relevant parameters, the quantum interaction parameter  $\Gamma/Z$ ,  $n/\tilde{n}$  etc... From Eq. (98) we observe that the traditional logarithmic term associated with the ideal and weakly nonideal classical plasmas is represented in our formula by  $Q = R_{\chi}(bq_1)(1+G(T))$ . Since  $G(T) \ll 1$  in the range of densities of interest, Eqs. (94) and (95) permit to express  $Q$  in the form

$$Q(n/\tilde{n}) \approx 1 - C_0(n/\tilde{n})^{1/3} + C_1(n/\tilde{n})^{2/3} + \dots, \quad bq_1 < 2\pi, \quad (119)$$

and

$$Q(n/\tilde{n}) \approx B_0(n/\tilde{n})^{-4/3} - 4 \exp[-A_0(n/\tilde{n})^{1/3}] [A_1 + A_2(n/\tilde{n})^{1/3} + A_3(n/\tilde{n})^{2/3} + A_4(n/\tilde{n}) + A_5(n/\tilde{n})^{4/3}] + \dots, \quad bq_1 > 2\pi, \quad (120)$$

where the constants  $A_0, A_1, B_0$  etc. . . . are defined in Appendix D.

In dimensionless form, Eq. (98) becomes,

$$\sigma_Q/\omega_p = \sigma_Q^*(n/\tilde{n}) (d_F/\bar{\lambda}_e) (n/\tilde{n})^{1/3} (\Gamma/Z)^{-1}, \quad (121)$$

where

$$\sigma_Q^*(n/\tilde{n}) = 3^{1/3} (\kappa_1/Z) (3/2\pi)^{5/6} / Q(n/\tilde{n}). \quad (122)$$

$\bar{\lambda}_e$  is given by Eq. (41),  $d_F$  is the Thomas-Fermi screening length,

$d_F = [(\hbar^2/4me^2)(\pi/3)^{1/3}]^{1/2}$  given numerically with  $n/\tilde{n}$  in Eq. (99), and

$$d_F/\bar{\lambda}_e = 12.30 n^{-1/6} T^{1/2}, \quad \Gamma/Z = 1.884 \times 10^8 n^{-1/3}. \quad (123)$$

By Eq. (115),  $\sigma_Q/\omega_p$  decreases with increasing interaction parameter  $\Gamma/Z$ .

Since  $\Gamma/Z$  is independent of the temperature  $T$  and decreases with increasing  $n$  like  $\Gamma/Z \sim n^{-1/3}$ , the nonideal quantum plasma ( $n > \tilde{n}$ ) becomes more ideal with increasing density  $n$  in contrast to the classical nonideal plasma ( $n < \tilde{n}$ ), which behaves

more "nonideal" as  $n$  increases since  $\gamma/Z \sim n^{1/3}$ . Consequently, the electrical conductivity of nonideal plasmas ( $n > \hat{n}$ ) has a minimum as a function of the pressure (or density  $n$ ) at some pressure  $\hat{p}$  (or density  $\hat{n}$ ). Along these lines, similar conclusions have been reported in experiments on the electrical conductivity of alkali plasmas<sup>3,30</sup>). This behavior should not be attributed only to the exponential electron density increase (with increasing pressure) due to (nonideal) ionization potential lowering, but also to the quantum effects ( $n > \hat{n}$ ) resulting from electron shell overlapping, electron tunneling, and electron transport in an ordered liquid-like ionized medium. All these effects contribute to the minimum observed in electrical conductivity data. The behavior of the electrical conductivity derived for nonideal quantum plasmas in Eq. (98), is shown in Fig. 4, which gives  $\sigma_Q$  versus the interaction parameter  $\Gamma/Z$ . A numerical illustration of the order of magnitude of  $\sigma_Q$  [Eq. (98)] is shown in Table III for a typical temperature  $T=10^4^\circ\text{K}$  at several densities of the degenerate electrons of a hydrogen plasma.

TABLE III. Electrical Conductivity of Quantum  
Hydrogen Plasma at  $T=10^4^\circ\text{K}$  ( $Z=1$ ).

$n[\text{cm}^{-3}]$	$10^{22}$	$10^{23}$	$10^{24}$
$\sigma_Q[\text{mho-cm}^{-1}]$	$7.240 \times 10^1$	$7.870 \times 10^2$	$9.480 \times 10^3$

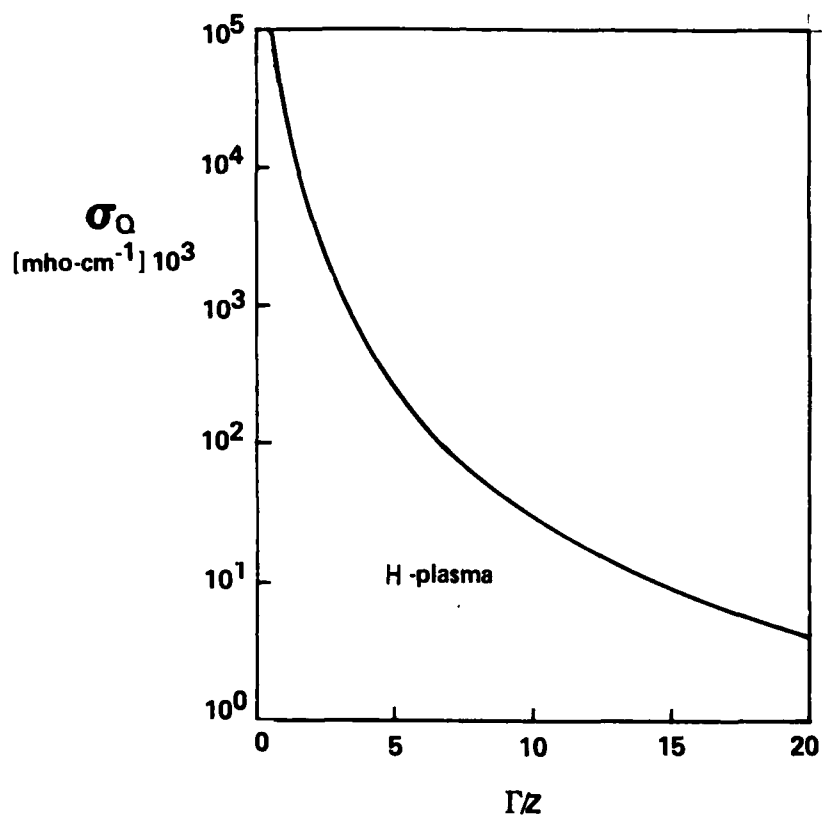


Fig. 4. Electrical Conductivity [mho-cm<sup>-1</sup>] of Quantum Nonideal Plasma Versus the Interaction Parameter  $\Gamma/Z$  at  $T=10^4$ °K.

# APPENDIX

## A. TRANSFORMATIONS.

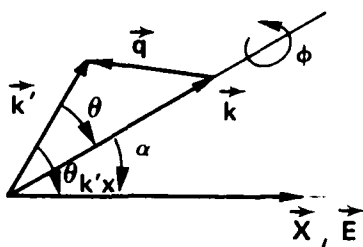
The integration over  $\vec{k}'$  in equation (32) is transformed to an integration over  $E'$  by the following change of variables

$$E' = \frac{\hbar^2 k'^2}{2m}, \quad k'^2 dk' = \frac{\sqrt{2}}{3} m^{3/2} \sqrt{E'} dE' \quad [A1]$$

The square bracket of Eq.(32), defining  $\phi$ , transforms to

$$1 - \frac{\phi'}{\phi} = 1 - \frac{k'_x C(E')}{k_x C(E)} \quad [A2]$$

where  $k'_x$  and  $k_x$  are related to  $\vec{k}'$  and  $\vec{k}$ . In accordance with the scattering diagram we have



$$\cos \alpha = \frac{k_x}{k}, \quad k'_x = k' \cos \theta_{k'_x}, \quad \cos \theta_{k'_x} = \cos \alpha \cos \theta + \sin \alpha \cos(\phi_x - \phi), \quad [A3]$$

With this change of variables, the integrals in Eq.(32) are over the energy  $E'$ , the angle  $\theta$  and the azimuthal angle  $\phi$ , where the latter integration over  $\phi$  cancels the term containing  $\cos(\phi_x - \phi)$ , since for each interaction we have a constant  $\vec{q}$ .

Hence,

$$d^3 \vec{k}' = 2\pi \sin \theta d\theta \frac{\sqrt{2} m^{3/2}}{\hbar^3} \sqrt{E'} dE', \quad [A4]$$

Considering the triangle  $(\vec{k}, \vec{k}', \vec{q})$  in the above diagram indicates that

$$q^2 = k'^2 + k^2 - 2kk' \cos \theta, \quad [A5]$$

where

$$|\vec{k}'| = |\vec{k}|, \quad (1 - \cos \theta) = \frac{q^2}{2k^2}, \quad \sin \theta d\theta = \frac{q dq}{k^2}. \quad [A6]$$

## B. INTEGRAL $I_e$

The integral in Eq.(35) is of the form

$$I_e = (\delta_e / \pi)^4 \int_0^{\hat{q}_e} \frac{q^5 dq}{(1 + a^2 q^2)^{1/2} [\text{Exp}[\frac{\hbar\omega_p}{KT} (1 + a^2 q^2)^{1/2}] - 1]} \quad [B1]$$

Let  $\epsilon = \frac{\hbar\omega_p}{KT} (1 + a^2 q^2)^{1/2}$  and  $\epsilon_p = \frac{\hbar\omega_p}{KT}$ , then  $I_e$  becomes

$$I_e = \frac{\delta_e^4 \epsilon_p}{\pi^4 (a\epsilon_p)^6} \int_{\epsilon_p}^{\hat{\epsilon}_e} \frac{(\epsilon^2 - \epsilon_p^2)^2 d\epsilon}{e^\epsilon - 1} \quad [B2]$$

Eq.[B2] contains integrals of Debye<sup>32)</sup> type which are given by

$$\int_0^x \frac{t^n dt}{e^t - 1} = x^n \left[ \frac{1}{n} - \frac{x}{2(n+1)} + \sum_{v=1}^{\infty} \frac{B_{2v} x^{2v}}{(2v+n)(2v)!} \right] \quad [B3]$$

with  $n = 4, n=2$ , and  $n=0$ . For  $n=0$  Eq. [B3] has a logarithmic solution and

for  $n \ll \tilde{n}$  we have  $\epsilon_p \ll 1$ ,  $\epsilon_e \ll 1$ ,  $a^2 \hat{q}_e^2 > 1$ . Thus,  $I_e$  becomes

$$I_e = \frac{4}{a^2 \epsilon_p} R_e(\epsilon_p, a\hat{q}_e) \quad [B4]$$

where

$$\begin{aligned} R_e(\epsilon_p, a\hat{q}_e) = & 1 - \frac{2}{5} \hat{\epsilon}_e + 4 \sum_{v=1}^{\infty} \frac{B_{2v} \hat{\epsilon}_e^{2v}}{(2v+4)2v!} - \frac{1}{(a\hat{q}_e)^4} \left[ 1 - \frac{2}{5} \epsilon_p + 4 \sum_{v=1}^{\infty} \frac{B_{2v} \epsilon_p^{2v}}{(2v+5)2v!} \right] \\ & - \frac{4}{(a\hat{q}_e)^2} \left[ 1 - \frac{\hat{\epsilon}_e}{3} + 2 \sum_{v=1}^{\infty} \frac{B_{2v} \epsilon_e^{2v}}{(2v+2)2v!} \right] + \\ & \frac{4}{(a\hat{q}_e)^4} \left[ 1 - \frac{\epsilon_p}{3} + 2 \sum_{v=1}^{\infty} \frac{B_{2v} \epsilon_p^{2v}}{(2v+2)(2v)!} \right] + \frac{4}{(a\hat{q}_e)^4} \left[ \ln \left\{ \frac{\hat{\epsilon}_e}{\epsilon_p} - 1 \right\} + \epsilon_p - \hat{\epsilon}_e \right] \quad [B5] \end{aligned}$$

It is seen that  $R_e(\epsilon_p, a\hat{q}_e) \approx 1$  for  $\hat{\epsilon}_e \ll 1$ , i.e. for  $n \ll \tilde{n}$ .

### C. Expansion of C(E)

Introducing an integral operator L representing the integrals over  $\vec{k}'$ , Eq. (55) can be formally written as

$$L(v_x C(E)) = -v_x \frac{\partial f_o}{\partial E}, \quad [C1]$$

where the function C(E) is to be expanded in a power series of (E -  $\xi$ ). Define

$$\psi = v_x C(E) = v_x \sum_{\mu} C_{\mu} (E - \xi)^{\mu}. \quad [C2]$$

In order to determine the coefficients of the series  $C_{\mu}$ , we use the Kohler variational principle<sup>33)</sup>, which makes use of the expression

$$(\psi, L\psi) = \iiint \psi (L\psi) d^3 \vec{k}, \quad [C3]$$

and

$$(\psi, -v_x \frac{\partial f_o}{\partial E}) = -\iiint v_x \psi \frac{\partial f_o}{\partial E} d^3 \vec{k}, \quad [C4]$$

where the integrations are carried over a constant energy surface of the electron before the interaction. The operator L is defined by Eq. (55) with the integration over  $\vec{k}'$ , and  $\psi$  is the series [C2]. Furthermore,

$$(\psi, L\psi) = \sum_{\mu, \nu} C_{\mu} C_{\nu} D_{\mu\nu} \quad [C5]$$

and

$$(\psi, -v_x \frac{\partial f_o}{\partial E}) = \sum_{\mu} C_{\mu} N_{\mu}. \quad [C6]$$

with

$$D_{\mu\nu} = \frac{1}{(2\pi)^3 K T} \iint v_x (E - \xi)^{\mu} \bar{W}(\vec{k}', \vec{k}) [v_x (E - \xi)^{\nu} - v'_x (E' - \xi)^{\nu}] \cdot d^3 \vec{k} d^3 \vec{k}' \quad [C7]$$

and

$$N_{\mu} = - \int v_x^2 \frac{\partial f_o}{\partial E} (E - \xi)^{\mu} d^3 \vec{k}, \quad [C8]$$

We seek the maximum of  $(\psi, L\psi)$  with the supplementary condition<sup>22)</sup>,

$$(\psi, L\psi) = (\psi, -v_x \frac{\partial f_o}{\partial E}) \quad [C9]$$

or

$$\sum_{\mu, \nu} C_{\mu} C_{\nu} D_{\mu\nu} = \sum_{\mu} C_{\mu} N_{\mu}, \quad [C10]$$

by Eqs.[C5] and [C6]. For this purpose we add the constraint [C9] to Eq.[C6] multiplied by a Lagrangian parameter  $\lambda$  and obtain the maximum from the condition

$$0 = \frac{d}{dC_\mu} \sum_{\mu\nu} C_\mu C_\nu D_{\mu\nu} + \lambda \sum_{\mu} C_\mu N_\mu = 2 \sum_{\nu} C_\nu D_{\mu\nu} + \lambda N_\mu \quad [C11]$$

Multiplication by  $C_\mu$  and a summation over  $\mu$  yields by comparison with Eq. [C10]

$\lambda = -2$ . Hence, Eq.[C11] reduces to the system of equations

$$\sum_{\nu} C_\nu D_{\mu\nu} = N_\mu \quad [C12]$$

which determine  $C_\nu$ . In particular  $C_o = \frac{N_o}{D_{oo}}$ .

#### D. Evaluation of $L(\gamma/Z, n/\tilde{n})$

In accordance with Eq.(54)  $L(\gamma/Z, n/\tilde{n}) = \kappa_i R_e + Z\kappa_e R_i$  where  $R_e$  and  $R_i$  are given by Eqs. [B5] and (47) respectively. By Eqs. (100) - (103),  $R_e(\gamma/Z, n/\tilde{n})$  becomes for  $n < \tilde{n}$ :

$$\begin{aligned} R_e(\gamma/Z, n/\tilde{n}) = & 1 + \mu_1 (n/\tilde{n})^{1/3} [1 + a_0 (\gamma/Z)^{-1}]^{1/2} (\gamma/Z)^{1/2} \\ & + \{a_1 + \mu_2 (n/\tilde{n})^{2/3} [1 + a_0 (\gamma/Z)^{-1}]\} (\gamma/Z)^1 \\ & + \mu_3 (n/\tilde{n})^{1/3} [1 + a_0 (\gamma/Z)^{-1}]^{1/2} (\gamma/Z)^{3/2} \\ & + \{a_2 + \mu_4 (n/\tilde{n})^2 [1 + a_0 (\gamma/Z)^{-1}]\} (n/\tilde{n})^{2/3} [1 + a_0 (\gamma/Z)^{-1}] (\gamma/Z)^2 + \dots \end{aligned} \quad [D1]$$

where

$$a_0 = (4\pi/3)^{2/3} \pi \kappa_e, \quad a_1 = -(36)^{1/3} / \pi^{5/3} \kappa_e, \quad a_2 = -4(18\pi)^{1/3} / \kappa_e \quad [D2]$$

$$\begin{aligned} \mu_1 &= -4\pi(32)^{1/6}/5, \\ \mu_2 &= 2^{8/3} \pi^2 / 9, \\ \mu_3 &= 4\sqrt{2} / (3\pi^2)^{1/3} \kappa_e, \\ \mu_4 &= -\pi^4 / 90 \end{aligned} \quad [D3]$$

In accordance with Eqs. (47) and (104)  $R_i(\gamma/Z, n/\tilde{n})$  is given for  $n < \tilde{n}$  by:

$$\begin{aligned} R_i(\gamma/Z, n/\tilde{n}) = & 1 - b_0 (n/\tilde{n})^{1/3} [1 + Z_0 (\gamma/Z)^{-1/2}]^{-1} + \sum_{v=1}^{\infty} b_v (n/\tilde{n})^{2v/3} \\ & \cdot [1 + Z_0 (\gamma/Z)^{-1/2}]^{-2v} \end{aligned} \quad [D4]$$

where

$$b_0 = (4/5)(2\pi)^{3/2} (\pi/3Z)^{1/3} \kappa_i^{1/2} (m/M)^{1/2} \quad [D5]$$

$$b_v = 4B_{2v} (5b_0/2)^{2v/3} / (2v+2)2v! \quad [D6]$$

Based on the definition in section VI,  $Q(n/\tilde{n}) = R_i(1+G(T)) \approx R_i(n/\tilde{n})$  for  $G(T) \ll 1$ . In accordance with Eqs. (94), (95) and (8) with  $\gamma/Z \gg 1$ ,  $R_i(n/\tilde{n})$  is given by:

$$R_i(n/\tilde{n}) = 1 - C_0 (n/\tilde{n})^{1/3} + 4 \sum_{v=1}^{\infty} C_v (n/\tilde{n})^{2v/3}, \quad b\hat{q}_i < 2\pi, \quad [D7]$$

where

$$C_0 = (8/5) \pi (2\pi \kappa_i)^{1/2} (3Z)^{-1/3} (m/M)^{1/2}, \quad [D8]$$

$$C_v = B_{2v} (5C_0/2)^{2v/3} / (2v+2) 2v! \quad [D9]$$

$B_{2v}$  are Bernoulli numbers and  $v$  is an index of summation. Furthermore,

$$R_{\chi}(n/\bar{n}) = B_0 (n/\bar{n})^{-4/3} - 4 \sum_{v=1}^{\infty} \exp[-v A_{v0} (n/\bar{n})^{1/3}] \left\{ A_{v1} + A_{v2} (n/\bar{n})^{1/3} + A_{v3} (n/\bar{n})^{-2/3} + A_{v4} (n/\bar{n})^{-1} + A_{v5} (n/\bar{n})^{-4/3} \right\}, \quad b\hat{q}_1 > 2\pi, \quad [D10]$$

where

$$B_0 = 96\zeta(5) / (5C_0/2)^4, \quad [D11]$$

$$\begin{aligned} A_{v0} &= 5C_0/2, \\ A_{v1} &= 1/v, \\ A_{v2} &= 4(5C_0/2)^{-1}/v^2, \\ A_{v3} &= 12(5C_0/2)^{-2}/v^3, \\ A_{v4} &= 24(5C_0/2)^{-3}/v^4, \\ A_{v5} &= 24(5C_0/2)^{-4}/v^5. \end{aligned} \quad [D12]$$

# REFERENCES

1. W. Ebeling and G. Roepke, Ann. Physik 36, 429 (1979).
2. V. S. Rogov, Teplofiz. Vys. Temp. 8, 689 (1970).
3. S. G. Barolskii, N. V. Yermokin, P. P. Kulik, and V. A. Riabii, Teplofiz. Vys. Temp. 114, 702 (1976).
4. M. Skowronek, J. Rons, A. Goldstein, and F. Cabannes, Phys. Fluids 13, 378 (1970).
5. G. E. Norman and A. A. Valuev, Teplofiz. Vys. Temp. 15, 191 (1977).
6. R. Roepke and K. Gunter, Plasma Physik 15, 299, (1975).
7. W. Ebeling, W. D. Kraft, and D. Kremp, in: Phenomena in Ionized Gases (Berlin 1977).
8. R. Landshoff, Phys. Rev., 76, 904 (1949).
9. R. S. Cohen, L. Spitzer, and P. M. Routhly, Phys. Rev. 80, 230 (1950)
10. H. E. Wilhelm, Can. J. Phys. 51, 2604 (1973).
11. L. Spitzer, Physics of Fully Ionized Gases (Interscience, New York 1956).
12. S. G. Barolskii, N. V. Ermokhin, P. P. Kulik, and V. M. Melnikov, Sov. Phys. JETP 35, 94 (1972).
13. Yu. V. Ivanov, V. B. Mintsev, V. E. Fortov, and A. N. Dremin, Sov. Phys. JETP 44, 112 (1976).
14. C. Goldbach, G. Nollez, S. Popovic, and M. Popovic, Z. Naturforsch. 33a, 11 (1977).
15. N. N. Iermohin, B. M. Kovaliov, P. P. Kulik, and V. A. Ryabyi, J. Physique 39, Supp. 5, 200 (1978).
16. H. E. Wilhelm, Phys. Rev. 187, 383 (1969).
17. H. E. Wilhelm, in press (1981).
18. Ref. 14. p. 16, Eq. (11).
19. Yu. L. Klimontovich, Sov. Phys. JETP 35, 920 (1972).
20. Yu. L. Klimontovich, Statisticheskeya Teoriya Neobratimymykh Prozessor (Moscow, Nauka, 1975).
21. D. N. Zubarew, Statistische Thermodynamik des Nichtgleichgewichts (Berlin 1976).

22. A. Haug, "Theoretical Solid State Physics" (Pergamon Press, New York 1972).
23. R. Kubo and T. Nagamiya, "Solid State Physics" (McGraw Hill, New York 1969).
24. N. E. Cusack, Contemp. Phys. 8, 583 (1967).
25. A. Sommerfeld and H. Bethe, Electron Theory of Metals, Handbook of Physics, vol. 24/2 (Springer, Berlin 1933).
26. A. I. Akhiezer et al., "Plasma Electrodynamics", Non-linear Theory and Fluctuations vol. 2 (Pergamon Press, New York 1975).
27. L. I. Schiff, "Quantum Mechanics" (McGraw-Hill, New York 1955).
28. L. J. Shaw and J. M. Ziman, "Solid State Physics" vol. 15, p. 221 (Acad. Press, New York 1963).
29. T. N. Batanova, B. M. Kovalev, P. P. Kulik, and V. A. Ryabyi, Teplofiz. Vys. Temp. Vol. 15, 643 (1976).
30. S. G. Barolskii et al, 12th ICPIG, Eindhoven, 1975, p. 181 and p. 184.
31. A. G. Sitenko, Electromagnetic Fluctuations in Plasmas (Acad. Press, New York 1967).
32. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, (Dover Publ. New York 1965).
33. M. Kohler, Z. Physik 125, 679 (1949).
34. Yu. L. Klimontovich and V. P. Silin, Usp. Fiz. Nauk 70, 247 (1960).
35. I. S. Gradshteyn and I. W. Ryzhik, Table of Integrals Series and Products (Acad. Press, New York 1965).
36. I. Ya. Dikhter, V. A. Zeigarnik, and S. V. Smagin Teplofiz. Vys. Temp. 17, 256 (1979).
37. A. A. Valuev and G. E. Norman Teplofiz. Vys. Temp. 15, 689 (1977).

## V. ANOMALOUS DIFFUSION ACROSS MAGNETIC FIELDS IN PLASMAS

By

H. E. Wilhelm

### Abstract

The anomalous diffusion transverse to a homogeneous magnetic field  $\vec{B}_0$  resulting from the interaction of the charged particles with the electric microfields in plasmas with an approximate local thermal equilibrium is analyzed by means of statistical methods based on the Langevin equation. The correlation functions of the stochastic velocity and electric microfields are calculated in closed form, from which an anomalous transverse diffusion coefficient  $D_{\perp} = (kT/m)/(3/\sqrt{2})|\omega|$  and momentum relaxation time  $\tau = (\sqrt{2}|\omega|)^{-1}$  are derived for particles of charge  $e \lesssim 0$ , mass  $m$ , and gyration frequency  $\omega = eB_0/m$  ( $kT$  = thermal energy). Comparison with the phenomenological Bohm diffusion coefficient  $D_{\perp}^B = kT/16|e|B_0$  indicates that anomalous diffusion in nonturbulent plasmas is considerably stronger than in turbulent plasmas.

## INTRODUCTION

According to the kinetic theory (binary interactions) of plasmas<sup>1)</sup>, the charged particles (electrons or ions) should diffuse across an external magnetic field  $\vec{B}_0$  with a diffusion coefficient  $D_{\perp} = (kT/m)\tau/(1 + \omega^2\tau^2)$ , where  $\tau$  is their momentum relaxation time and  $\omega = eB_0/m$  is their gyration frequency ( $kT$  = thermal energy,  $m$  = mass,  $e \lesssim 0$  = charge of particles). The dependence  $D_{\perp} \propto B_0^{-2}$  on the induction has been observed for  $\omega^2\tau^2 \gg 1$ , e.g., in weakly ionized plasmas in which the charged particles interact mainly with the neutrals.<sup>2)</sup> For plasmas with predominant Coulomb interactions, the experiments frequently indicate an anomalous diffusion  $D_{\perp} \propto B_0^{-1}$ , e.g., for low pressure arcs<sup>3)</sup>, hollow cathode discharges of ion thrusters<sup>4)</sup>, magnetically contained fusion plasmas<sup>5)</sup>, magnetically insulated diodes<sup>6)</sup>, and magnetically insulated ion beams<sup>7)</sup>. Bohm was the first to investigate anomalous diffusion across magnetic fields in low pressure mercury arcs and derived from the experimental data the transverse diffusion coefficient<sup>8)</sup>

$$D_{\perp}^B = kT/16|e|B_0$$

Following the original suggestion by Bohm that the anomalous diffusion is caused by turbulent particle transport across the magnetic field lines, some not quite successful attempts have been made at explaining "Bohm diffusion" within the frame of plasma turbulence theory<sup>9,10)</sup>. Except for the dimensionless phenomenological coefficient  $C = 1/16$ , the Bohm formula can be readily deduced by means of dimensional analysis. More recent experiments indicate that  $1/20 < C < 1/2$ , depending on the type of plasma and the level of turbulence present<sup>11,12)</sup>.

We consider herein nonturbulent plasmas in a homogeneous magnetic field, which are fully ionized or contain neutrals only in small concentrations so that their interactions with the charged particles are negligible. For the diffusion processes the usual assumption is made that they perturb the Maxwell distributions of the electrons and ions only slightly. Based on the Langevin equations<sup>13,14)</sup> for the electrons and ions in a homogeneous  $\vec{B}_0$ -field, the correlation function for their

stochastic velocity fields and for the random electric microfields are calculated. The thermal equilibrium fluctuations of the electric microfield are shown to produce random particle drifts across  $\vec{B}_0$ , which result in an anomalous diffusion coefficient  $D_{\perp} \propto B_0^{-1}$ . The anomalous diffusion coefficient is of the magnitude of the maximum diffusion coefficient in a magnetic field, i.e.  $D_{\perp} > D_{\perp}^B$ .

Bohm diffusion in turbulent plasmas can be treated in an analogous manner. The evaluation of the correlation functions of the macroscopic velocity and electric field fluctuations in turbulent plasmas requires, however, different mathematical methods.

# THEORETICAL FOUNDATIONS

The stochastic motion  $\vec{v}(t)$  of a charged test particle ( $m, e$ ) under the influence of the random fluctuations  $\vec{E}(t)$  of the electric microfield of a quasi-homogeneous plasma is commonly described by the Fokker-Planck equation or the corresponding Langevin equation<sup>13,14</sup>. Within this theoretical approach, the diffusion coefficients perpendicular ( $D_{\perp}$ ) and parallel ( $D_{\parallel}$ ) to an external homogeneous magnetic field  $\vec{B}_0 = (0, 0, B_0)$  are given in terms of the mean square particle displacements at time  $t$  by<sup>13,14</sup>

$$D_{\perp} = \langle \Delta x^2(t) \rangle / 2t, \quad \langle \Delta x(t)^2 \rangle = \langle \Delta y(t)^2 \rangle; \quad D_{\parallel} = \langle \Delta z(t)^2 \rangle / 2t. \quad (1)$$

Eq. (1) contains the assumption that the plasma is isotropic in all planes perpendicular to  $\vec{B}_0$  ( $D_x = D_y = D_{\perp} \neq D_{\parallel}$ ). The particle displacements  $\Delta \vec{r}(t) = \int_t \vec{v}(t) dt$  are determined by the Langevin equation for the magnetoactive plasma<sup>13,14</sup>:

$$d\vec{v}_{\perp}(t)/dt = \frac{e}{m} [\vec{E}_{\perp}(t) + \vec{v}_{\perp}(t) \times \vec{B}_0] - \frac{1}{\tau} \vec{v}_{\perp}(t) \quad (2)$$

$$d\vec{v}_{\parallel}(t)/dt = \frac{e}{m} E_{\parallel}(t) - \frac{1}{\tau_{\parallel}} \vec{v}_{\parallel}(t) \quad (3)$$

In Eqs. (2) and (3),  $\vec{E}(t)$  is the stochastic longitudinal (nonrelativistic) microfield produced collectively by the charged field particles (electrons and ions) of the thermal equilibrium plasma at the position of the test particle ( $m, e$ ), whereas  $\vec{v}(t) \times \vec{B}_0$  is the stochastic Lorentz field generated by the random motion  $\vec{v}(t)$  of the test particle charge across the field lines  $\vec{B}_0$ . The averages of the stochastic fields for an ensemble of test particles vanish,

$$\langle \vec{v}(t) \rangle = \vec{0}, \quad \langle \vec{E}(t) \rangle = \vec{0} \quad (4)$$

The ensemble averages over the stochastic fields are time-independent, since they are identical with the time averages over periods  $t \gg \tau, \tau_{\parallel}$ , in statistical equilibrium. These averages are calculated by means of the velocity distribution of the test particles, which is a Gauss or Maxwell distribution (Markoff process) for times  $t \gg \tau, \tau_{\parallel}$ .

Accordingly,

$$\langle v_i^2(t) \rangle = kT/m, \quad i = x, y, z. \quad (5)$$

The test particle experiences a friction force  $-m(\dot{v}_\perp(t)/\tau + \dot{v}_\parallel(t)/\tau_\parallel)$  as it moves through the random impulses from the fluctuating microfield  $\vec{E}(t)$ , i.e.,  $\tau = \tau\{\vec{E}_\perp\}$  and  $\tau_\parallel = \tau_\parallel\{\vec{E}_\parallel\}$  are integral functionals of  $\vec{E}(t)$ . Since the transport mechanisms in the directions perpendicular and parallel to  $\vec{B}_0$  involve and are free from induction, respectively,  $\tau$  and  $\tau_\parallel$  are necessarily different.

In accordance with the theory of the Langevin equation, the velocity  $\vec{v}(t)$  of the test particle changes significantly during a relaxation time  $\tau$ , but may vary only by a fraction of the change  $|\vec{v}(t + \tau) - \vec{v}(t)|$  during a single field impulse  $\vec{E}(t)$ .<sup>13,14)</sup> In rarefied plasmas this condition is satisfied if the plasma frequency  $\omega_p \sim \omega_E$  is large compared with the relaxation frequency ( $\epsilon_0$  = dielectric permittivity of vacuum,  $e_0$  = elementary charge,  $m_0$  = electron mass)<sup>14)</sup>

$$\omega_p \gg \tau^{-1}, \tau_\parallel^{-1}, \quad \omega_p = (ne_0^2/\epsilon_0 m_0)^{1/2} \quad (6)$$

For the Brownian motion of a macroscopic test particle (colloid) in a viscous fluid, the relaxation time  $\tau$  is given by the Einstein-Stokes formula. For the stochastic motion of a microscopic particle, the relaxation frequency  $\tau^{-1}$  has to be calculated as a correlation integral of the microfield. The fascinating idea of the interrelation between relaxation time and microfield has been first formulated by Einstein in his investigation on the thermal equilibrium between atoms and Planck radiation.<sup>15)</sup>

# ANOMALOUS DIFFUSION

We consider a fully ionized plasma of  $n(\vec{r})$  electrons ( $e = -e_0$ ,  $m = m_0$ ) and  $n(\vec{r})/Z$  ions ( $e = Ze_0$ ,  $m = M$ ) per unit volume in a homogeneous magnetic field  $\vec{B}_0 = (0, 0, B_0)$ . A weak density gradient  $\nabla n(\vec{r})$  is assumed to be present so that i) the plasma can be considered to be statistically isotropic in planes perpendicular to  $\vec{B}_0$  and ii) the associated diffusion currents perturb the local thermal equilibrium only slightly. The Langevin equations (2) and (3) represent two coupled differential equations for the transverse components  $v_x(t)$  and  $v_y(t)$  and one independent differential equation for the axial component  $v_z(t)$  of the velocity fluctuation of the test particle, which have the formal solutions:

$$v_x(t) = (u_0 \cos \omega t + v_0 \sin \omega t) e^{-t/\tau} + (e/m) \int_0^t e^{-(t-s)/\tau} [E_x(s) \cos \omega(t-s) + E_y(s) \sin \omega(t-s)] ds, \quad (7)$$

$$v_y(t) = (v_0 \cos \omega t - u_0 \sin \omega t) e^{-t/\tau} + (e/m) \int_0^t e^{-(t-s)/\tau} [E_y(s) \cos \omega(t-s) - E_x(s) \sin \omega(t-s)] ds, \quad (8)$$

$$v_z(t) = w_0 e^{-t/\tau_{||}} + (e/m) \int_0^t E_z(s) e^{-(t-s)/\tau_{||}} ds, \quad (9)$$

where

$$\omega = eB_0/m \lesssim 0, \quad e \lesssim 0, \quad (10)$$

and  $(u_0, v_0, w_0) = \vec{v}(t=0)$  is the initial value of the stochastic field. Scalar multiplication of the solution vector  $\vec{v}(t)$  with the initial vector  $\vec{v}(t=0)$  and averaging yields, after generalizing the resulting correlation functions for "initial" times  $t'$ , in the limit  $t \gg \tau, \tau_{||}$ :

$$\langle v_x(t) v_x(t') \rangle = \langle v_y(t) v_y(t') \rangle = (kT/m) \cos \omega(t-t') e^{-|t-t'|/\tau}, \quad (11)$$

$$\langle v_z(t) v_z(t') \rangle = (kT/m) e^{-|t-t'|/\tau_{||}}, \quad (12)$$

since  $\langle u_0^2 \rangle = \langle v_0^2 \rangle = \langle w_0^2 \rangle = kT/m$ . The mean square displacements in the directions perpendicular and parallel to  $\vec{B}_0$  are proportional to  $t$ ,

$$\langle \Delta x(t)^2 \rangle = \langle \Delta y(t)^2 \rangle = \int_0^t \int_0^t \langle v_x(t) v_x(t') \rangle dt dt' = 2 \frac{kT}{m} \tau t / (1 + \omega^2 \tau^2), \quad (13)$$

$$\langle \Delta z(t)^2 \rangle = \int_0^t \int_0^t \langle v_z(t) v_z(t') \rangle dt dt' = 2 (kT/m) \tau_{||} t, \quad (14)$$

by Eqs. (11) and (12). Accordingly, the transverse and parallel diffusion coefficients defined in Eq. (1) are

$$D_{\perp} = (kT/m) \tau / (1 + \omega^2 \tau^2), \quad D_{||} = (kT/m) \tau_{||}. \quad (15)$$

$D_{||}$  and  $\tau_{||}$  are known from the diffusion theory for plasmas without magnetic fields<sup>16)</sup>.

For the evaluation of the transverse relaxation time  $\tau$  in  $D_{\perp}$ , the solution (7) is squared and averaged,

$$\begin{aligned} \langle v_x(t)^2 \rangle &= \langle (u_0 \cos \omega t + v_0 \sin \omega t)^2 \rangle e^{-2t/\tau} \\ &+ (e/m)^2 e^{-2t/\tau} \int_0^t \int_0^t e^{(r+s)/\tau} [\langle E_x(r) E_x(s) \rangle \cos \omega(t-r) \cos \omega(t-s) \\ &+ \langle E_y(r) E_y(s) \rangle \sin \omega(t-r) \sin \omega(t-s)] dr ds, \end{aligned} \quad (16)$$

since  $\vec{v}(0)$  and  $\vec{E}(t)$  are statistically independent and  $\langle E_i(r) E_j(s) \rangle = 0$  for  $i \neq j$ . Eq. (16) becomes after integration over the coherence strip by means of the transformation  $\xi = r - s$ ,  $\eta = (r + s)/2$ , in the limit  $t \gg \tau$ :

$$kT/m = \frac{1}{2} (e/m)^2 \tau \int_{-\infty}^{+\infty} \phi(\xi) \cos \omega \xi d\xi \quad (17)$$

where

$$\phi(\xi) \equiv \phi(r-s) = \langle E_x(r) E_x(s) \rangle = \langle E_y(r) E_y(s) \rangle \quad (18)$$

in view of the isotropy in planes  $\perp \vec{B}_0$ . Eq. (17) gives the relation between  $\tau$  and  $\vec{E}_\perp(t)$  through an integral over the microfield correlation function  $\phi(\xi)$ , which will be shown to decrease exponentially with increasing  $|\xi|$ .

The correlation function for  $\vec{E}_\perp(t)$  can be expressed in terms of the correlation functions for  $\vec{v}_\perp(t)$ , since these stochastic vector functions are interrelated through the Langevin equation. By Eq. (2),

$$\dot{\vec{E}}_\perp = -\vec{v}_\perp \times \vec{B}_0 + (m/e)\vec{v}_\perp/\tau + (m/e)d\vec{v}_\perp/dt \quad (19)$$

or

$$\dot{\vec{E}}_\perp = -\vec{v}_\perp \times \vec{B}_0 + (m/e)\vec{v}_\perp/\tau, \quad t \gg \tau, \quad (20)$$

as known from the general theory of the Langevin equation<sup>17)</sup>. Since the correct  $\langle \Delta x^2 \rangle$  and  $D_\perp$  are obtained in the limit  $t \gg \tau$  ( $d\vec{v}_\perp/dt \rightarrow 0$ )<sup>17)</sup>, the correlation function of the microfield is directly calculated from Eq. (20) as

$$\begin{aligned} \langle E_x(t) E_x(t') \rangle &= B_0^2 \langle v_y(t) v_y(t') \rangle + (m/e\tau)^2 \langle v_y(t) v_y(t') \rangle \\ &\quad - (mB_0/e\tau) [\langle v_x(t) v_y(t') \rangle + \langle v_y(t) v_x(t') \rangle]. \end{aligned} \quad (21)$$

The method used for the determination of the  $xx$  and  $yy$  velocity correlations in Eqs. (11) and (12) yields, by means of the solutions (7) and (8), for the asymmetric velocity correlations

$$\langle v_x(t) v_y(t') \rangle = +(kT/m) \sin \omega(t-t') e^{-|t-t'|/\tau}, \quad (22)$$

$$\langle v_y(t) v_x(t') \rangle = -(kT/m) \sin \omega(t-t') e^{-|t-t'|/\tau}. \quad (23)$$

These weak correlations are caused by the rotation of the charges in the magnetic field, and vanish for  $\omega = eB_0/m \rightarrow 0$ . Substitution of Eqs. (11) - (12) and (22) - (23)

into Eq. (21) gives the desired correlation function for the microfield components  $E_x(t)$  or  $E_y(t)$ ,

$$\langle E_x(t) E_x(t') \rangle = (1 + \omega^2 \tau^2) (\omega \tau)^{-2} (kT/m) B_0^2 \cos \omega(t-t') e^{-|t-t'|/\tau}. \quad (24)$$

Combining of Eqs. (24), (17), and (18) results in an eigen-value equation for the relaxation time,

$$1 = \frac{1}{2} (1 + \omega^2 \tau^2) \tau^{-1} \int_{-\infty}^{+\infty} e^{-|\xi|/\tau} \cos^2 \omega \xi d\xi. \quad (25)$$

Since the integral is  $2\tau(1 + 2\omega^2 \tau^2)/(1 + 4\omega^2 \tau^2)$ , Eq. (25) has the solution

$$\omega^2 \tau^2 = 1/2; \quad |\omega| \tau = 1/\sqrt{2}. \quad (26)$$

This remarkable result gives for the microfield driven diffusion of charged particles (e,m) transverse to a magnetic field  $\vec{B}_0$  [Eq. (15)] the transport coefficients:

$$D_{\perp} = (kT/m) / [(3/\sqrt{2}) |\omega|], \quad \tau^{-1} = \sqrt{2} |\omega|. \quad (27)$$

For electrons ( $e = -e_0$ ,  $m = m_0$ ) and ions ( $e = Ze_0$ ,  $m = M$ ) at the temperature  $T$ , the anomalous transverse diffusion and relaxation frequencies are:

$$D_{\perp}^e = kT/(3/\sqrt{2}) e_0 B_0, \quad \tau_e^{-1} = \sqrt{2} e_0 B_0 / m_0, \quad (28)$$

$$D_{\perp}^i = kT/(3/\sqrt{2}) Ze_0 B_0, \quad \tau_i^{-1} = \sqrt{2} Ze_0 B_0 / M. \quad (29)$$

This completes the theory of the anomalous, microfield driven diffusion of charged particles across a homogeneous magnetic field in nonturbulent plasmas, which are in an approximate local thermodynamic equilibrium.

## DISCUSSION

The anomalous diffusion of charged particles across magnetic fields is due to the eigenvalue character of  $\omega\tau$  [Eq. (26)], which reflects the interrelation of electric and velocity field correlation [Eq. (24)]. The mathematical maximum of the transverse diffusion coefficient is obtained from Eq. (15),

$$dD_{\perp}/dt = (1 - \omega^2\tau^2)(1 + \omega^2\tau^2)^{-1}kT/m = 0, \quad (30)$$

as

$$\hat{D}_{\perp} = kT/2m|\omega|, \quad \tau^{-1} = |\omega| \quad (31)$$

The actual diffusion coefficient in Eq. (27) is somewhat smaller, but nearly equal to the mathematical maximum in Eq. (31),  $D_{\perp} \leq \hat{D}_{\perp}$  since  $3/\sqrt{2} = 2.121 \geq 2$ . Thus, in approximate thermal equilibrium, transverse diffusion is practically optimum.

Comparison of Eq. (27) with Bohm diffusion<sup>8)</sup> indicates that  $D_{\perp} \approx 8D_{\perp}^B$ . Although the phenomenological coefficient "8" is subject to experimental errors, it appears safe to conclude that transverse diffusion in plasmas is considerably weaker in presence of turbulence than in approximate thermal equilibrium.

The condition (6) for the applicability of the Langevin equation to electrons (e) and ions (i) becomes

$$(e) \quad \omega_p \gg \sqrt{2}|\omega_e|, \quad (i) \quad \omega_p \gg \sqrt{2}|\omega_i|, \quad (32)$$

by Eq. (26). These inequalities provide upper limits for the magnetic field intensity,

$$(e) \quad B_0 \ll (mn/2\epsilon_0)^{1/2}, \quad (i) \quad B_0 \ll (M/Z_m)(mn/2\epsilon_0)^{1/2}. \quad (33)$$

Eq. (32) or (33) corresponds for  $Z \sim 1$  to the (illustrative) condition that the r.m.s. induced field  $\langle (\vec{v}_{\perp s} \times \vec{B}_0)^2 \rangle^{1/2}$  of the random electron ( $s = e$ ) and ion ( $s = i$ ) is small compared with the r.m.s. electric microfield  $\langle \vec{E}_{\perp}^2 \rangle^{1/2}$ .

plasmas without magnetic fields,

$$(e) \quad (2kT/m)^{1/2} B_0 \ll \langle \vec{E}_i^2 \rangle^{1/2}, \quad (i) \quad (2kT/M)^{1/2} B_0 \ll \langle \vec{E}_i^2 \rangle^{1/2}, \quad (34)$$

where<sup>18)</sup>

$$\langle \vec{E}_i^2 \rangle = 2(1 + Z^{-1})nkT/\epsilon_0 \quad . \quad (35)$$

These inequalities, which can probably be relaxed from small ( $\ll$ ) to smaller ( $<$ ) in applications, are in general realized in plasmas in which anomalous diffusion is observed.

In experiments, the plasma is not always in an approximate local thermal equilibrium. Since the characteristic times for thermal relaxation  $T_e - T_i \rightarrow 0$  between electrons and ions and thermal anisotropy relaxation  $T_{i\perp} - T_{i\parallel} \rightarrow 0$  in the magnetic field are large compared with the momentum relaxation time ( $\tau$ ), transient plasmas may be encountered with  $T_{i\perp}^e \neq T_{i\perp}^i$ . In this nonequilibrium situation the diffusion formulas are still applicable if one sets  $T = T_{i\perp}^e$  in Eq. (28) and  $T = T_{i\perp}^i$  in Eq. (29).

Plasmas with diffusion in weak density gradients are stable and remain so even for larger density gradients due to the stabilizing effects of the homogeneous magnetic field<sup>19)</sup>. In more complicated plasma systems, e.g., low pressure arcs with current flow due to external electric fields, various convective instabilities<sup>19)</sup> and electron-ion streaming instabilities<sup>19)</sup> may arise. Anomalous diffusion in unstable and turbulent plasmas will be treated in a separate investigation.

## REFERENCES

1. I. P. Sharofsky, T. W. Johnston, M. P. Bachynski, *The Particle Kinetics of Plasmas* (Addison-Wesley, Reading 1966).
2. V. E. Golant, *Sov. Phys. - Uspekhi* 6, 161 (1963).
3. F. C. Hoh, *Rev. Mod. Phys.*, 34, 267 (1962).
4. H. R. Kaufman and R. S. Robinson, 14th International Electric Propulsion Conference, Princeton 1979.
5. F. Winterberg, *Phys. Rev.* 174, 212 (1968).
6. F. Winterberg, *Rev. Sci. Instr.* 43, 814 (1972).
7. F. Winterberg, *J. Plasma Phys.* 21, 301 (1979).
8. D. Bohm, *The Characteristics of Electrical Discharges in Magnetic Fields* (A. Guthrie and R. K. Wakerling, Eds., McGraw-Hill, New York 1949) pp. 12.
9. A. A. Vedenov and E. P. Velikhov, *Dokl. Acad. Nauk* 146, 65 (1962).
10. B. B. Kadomtsev and A. V. Nedospasov, *Plasma Phys.* C1, 230 (1960).
11. H. F. Rugge and R. V. Pyle, *Phys. Fluids* 7, 754 (1964).
12. H. R. Kaufman, Private Communication, 1 May 1981.
13. S. Chandrasekhar, *Rev. Mod. Phys.* 15, 1 (1943).
14. R. Becker, *Theory of Heat* (Springer, New York 1976).
15. A. Einstein, *Phys. Zs.* 18, 121 (1917).
16. I. Fidone, *Nuovo Cimento* 20, 1219 (1961).
17. A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York 1965).
18. H. E. Wilhelm, to be published 1981.
19. A. A. Vedenov, *Theory of Turbulent Plasma* (U. S. Department of Commerce, Springfield 1966).

## VI. COLLECTIVE MICROFIELD DISTRIBUTION IN THERMAL PLASMAS

By

H. E. Wilhelm

### ABSTRACT

The temperature and density dependent probability distribution  $W = W(\vec{E}; T, n)$  of the collective microfield  $\vec{E}$  in fully ionized, ideal plasmas is calculated from first principles of statistical mechanics. For typical ideal plasmas, the average microfield  $E_W = [12\pi(1 + Z^{-1})nKT]^{1/2}$  is by one to two orders of magnitude larger than the characteristic field (nearest neighbor approximation)  $E_H = E_H(n)$  of the Holtsmark microfield distribution  $P = P(E; n)$ . The Holtsmark theory and its later extensions are shown to be approximately valid for strongly nonideal plasmas only. The interrelations between (average) kinetic, interaction, collective microfield, and electric self energies is discussed. In particular, an equipartition  $\langle \vec{E}^2 / 8\pi \rangle = 3(1 + Z^{-1}) \times nKT/2$  among (average) microfield and kinetic particle energies in statistical equilibrium is derived by means of a thermodynamic model of plasma formation.

## INTRODUCTION

The probability distribution of the stochastic electric fields  $\vec{E}$  produced by the electrons and ions in random thermal motion, is of basic interest for plasma physics and for applications such as the evaluation of the transport properties, the Stark broadening of spectral lines, and the preionization of atoms. Holtsmark calculated the probability  $P(E)dE$  for the dimensionless electric microfield to be found with a magnitude between  $E$  and  $E + dE$  for a system of  $n/Z$  point charges  $Ze$  per unit volume with the result <sup>1)</sup>

$$P(E) = (2/\pi)E^{-1} \int_0^\infty x \sin x e^{-(x/E)^{3/2}} dx$$

where

$$E = |\vec{E}|/E_H, \quad E_H = 2\pi(4/15)^{2/3} Z^{1/3} e n^{2/3}$$

This integral functional of  $E$  shows the asymptotic behavior  $P(E) = (3/2)E^{-5/2}$  for  $E \rightarrow \infty$  so that already the second order moment (average field energy density) does not exist,  $\langle E^2 \rangle = \int_0^\infty P(E)E^2 dE = \infty$ ! Consecutively, Gans <sup>2)</sup> believed to have derived the correct distribution in form of an integral functional  $P(E)$  with converging moments by considering non zero radii  $r_0 < (4\pi n/3Z)^{-1/3}$  of the charged particles, which make zero distance approaches and infinite fields impossible. Both the Holtsmark and Gans distributions are independent of the temperature of the field particles, and describe essentially the fields of the nearest neighbor particles at an average distance  $\bar{r} = (4\pi n/3Z)^{-1/3}$  since  $E_H \approx e/\bar{r}^2$ . These theories are, therefore, approximately valid for strongly nonideal plasmas, in which the thermal energy  $KT$  is negligible compared with the average Coulomb interaction energy  $Ze^2/\bar{r}$  (electron-ion interactions). This conclusion has been confirmed experimentally by Vidal <sup>3)</sup>, who showed that observed Stark broadening by "cold" microwave discharge plasmas ( $Ze^2 n^{1/3} \gg KT$ ) is in good agreement with the Holtsmark or Gans distributions.

The later developments of microfield theory have been concerned with correcting and extending the Holtsmark theory under consideration of thermal effects and particle correlations. The formulation of a rigorous theory of the microfield in thermal plasmas of arbitrary nonideality is equivalent to the solution of the general many-particle Coulomb interaction problem, and has not been achieved yet. However, significant contributions to this problem were made with the help of collective coordinate and discrete particle methods by distinguished researchers, e.g., Broyles<sup>4,5)</sup>, Baranger and Mozer<sup>6,7)</sup>, Hettner and Wagner<sup>8,9)</sup>, and Hunger and Larenz<sup>10,11)</sup>.

Most laboratory plasmas, e.g., glow discharges, arc discharges at not more than atmospheric pressure, and thermonuclear fusion discharges, are ideal systems in which the Coulomb interaction energy is small compared to the thermal energy. An interesting counter example for a nonideal plasma is the ball lightning phenomenon, which consists of a highly ionized air plasma ( $n \sim 10^{18} \text{ cm}^{-3}$ ) of low temperature ( $T \sim 10^3 \text{ K}$ ). In ball lightning, the plasma appears to be in a highly viscous, quasi-liquid state due to the balance of thermal and (negative) Coulomb interaction energies,  $Ze^2 n^{1/3} \sim KT$ , the spherical shape and long life-time ( $\Delta t \sim 1 \text{ sec}$ ) being explainable by minimum energy considerations.<sup>12)</sup> In the following, we are concerned only with ideal plasmas, for which we derive the probability distribution of the collective electric microfields from first principles, i.e. the results are limited to interaction parameters

$$\gamma = Ze^2 n^{1/3} / KT = 1.670 \times 10^{-3} Z n^{1/3} T^{-1} \ll 1.$$

By the fundamental axiom of statistical mechanics of ideal systems in thermal equilibrium, all equilibrium distributions can be derived without consideration of the interactions which bring about the equilibrium.<sup>13)</sup> By extending this principle for many-particle systems with discrete energies to continuous media with random

energy densities  $u = \vec{E}(\vec{r}, t)^2 / 8\pi$ , we derive the probability distribution of the collective microfields  $\vec{E}(\vec{r}, t)$  in ideal plasmas without approximations. Previously, we have generalized the methods of statistical mechanics for hydrodynamic<sup>14)</sup> and plasma<sup>15)</sup> turbulence based on the generalized entropy principle for nonequilibrium systems.

# PROBABILITY DISTRIBUTION

Subject of the considerations is a homogeneous, fully ionized ideal plasma of volume  $\Omega$  containing  $n_e = n$  electrons and  $n_i = n/Z$  ions per unit volume. In thermal equilibrium, the kinetic energy densities of the electron and ion components are given by ( $m_s$  = particle mass)<sup>13)</sup>

$$\left\langle \sum_{\mu=1}^N m_s \vec{v}_{s\mu}^2 \right\rangle = \frac{3}{2} n_s \Omega K T, \quad N_s = n_s \Omega, \quad s = e, i. \quad (1)$$

During the random thermal motions of the charged particles, a continuous transformation of kinetic particle energy into potential electric energy occurs, and vice versa, due to the particle interactions through their longitudinal Coulomb fields (transverse electromagnetic interactions are negligible for  $m_s c^2 \ll K T$ ). By means of a thermodynamic model for the formation of a fully ionized plasma, we demonstrate that an equipartition of average random electric and kinetic energies exists in statistical equilibrium [Eq. (42)],

$$\langle \vec{E}^2 / 8\pi \rangle = \frac{3}{2} (1 + Z^{-1}) n K T \quad (2)$$

The electric field  $\vec{E}(\vec{r}, t)$  produced collectively by the electrons and ions at any point  $\vec{r} \in \Omega$  of the plasma and the field energy density  $u = \vec{E}(\vec{r}, t)^2 / 8\pi$  fluctuate with time  $t$  about the average values  $\langle \vec{E} \rangle = \vec{0}$  and  $\langle \vec{E}^2 / 8\pi \rangle \neq 0$ , respectively. The proposed problem is to derive the probability  $W(\vec{E}) d^3\vec{E}$  for finding the collective field fluctuation  $\vec{E}$  in the volume element  $d^3\vec{E} = dE_x dE_y dE_z$  about the point  $\vec{E} = (E_x, E_y, E_z)$  of the field space subject to the thermal equilibrium conditions (1) and (2).

In order to determine experimentally the collective microfield distribution  $W(\vec{E}) = W(\vec{E}^2 / 8\pi)$  in a homogeneous and isotropic plasma, one would have to measure the fluctuating field  $\vec{E}$  or the fluctuating energy density  $\vec{E}^2 / 8\pi$  in the vicinity  $\Delta^3\vec{r}$  of a fixed field point  $\vec{r} \in \Omega$  at consecutive times  $t_v = v\theta_v$ ,  $v = 1, 2, 3, \dots, N$  within experimental errors  $\Delta t_v \ll \theta_v$ , where  $\theta_v$  is a time interval which is large compared

with the correlation time of the stochastic field so that  $\langle \vec{E}(\vec{r}, t) \vec{E}(\vec{r}, t + \theta_v) \rangle = 0$  (within these limitations, the magnitude of  $\theta_v$  may be changed from one measurement  $v$  to the next  $v+1$ ). In a large number of such measurements,  $N \rightarrow \infty$ , the energy density  $\vec{E}_1^2/8\pi$  would be observed  $N_1$  times, ... , the energy density  $\vec{E}_\alpha^2/8\pi$  would be observed  $N_\alpha$  times, etc., where  $\vec{E}_\alpha^2/8\pi$  means an experimental value measured with an error  $\Delta(\vec{E}^2/8\pi)$ . The resulting step-shaped energy distribution  $N_\alpha = N_\alpha(\vec{E}_\alpha^2/8\pi)$  is represented by the partition

$$\begin{matrix} N_1 & N_2 & N_3 & \dots & N_\alpha & \dots & N_M \\ \vec{E}_1^2/8\pi & \vec{E}_2^2/8\pi & \vec{E}_3^2/8\pi & \dots & \vec{E}_\alpha^2/8\pi & \dots & \vec{E}_M^2/8\pi \end{matrix} \quad (3)$$

where

$$N_1 + N_2 + N_3 + \dots + N_\alpha + \dots + N_M = N, \quad (4)$$

$$N_1 \vec{E}_1^2/8\pi + N_2 \vec{E}_2^2/8\pi + N_3 \vec{E}_3^2/8\pi + \dots + N_\alpha \vec{E}_\alpha^2/8\pi + \dots + N_M \vec{E}_M^2/8\pi = N \langle \vec{E}^2/8\pi \rangle. \quad (5)$$

$N$  is the total number of measurements ( $N \rightarrow \infty$ ) and  $N \langle \vec{E}^2/8\pi \rangle$  is the total field energy density measured in the  $N$  independent observations. The entire energy density  $N \langle \vec{E}^2/8\pi \rangle$  can be distributed in a large number  $\Pi$  of ways over sets  $\{N_\alpha\}_N$  of numbers  $N_\alpha$ . By elementary combinatorics,<sup>13)</sup>

$$\Pi = N! / N_1! N_2! N_3! \dots N_\alpha! \dots N_M! \quad (6)$$

The energy distribution  $N_\alpha(\vec{E}_\alpha^2/8\pi)$  observed in statistical equilibrium is the most probable one. Thus,  $N_\alpha(\vec{E}_\alpha^2/8\pi)$  is determined by the condition for a maximum of i) the number  $\Pi$  of realizations or ii) the entropy  $S \sim \ln \Pi$ , subject to the constraints (4) and (5).

Accordingly, we determine the probability distribution  $N_\alpha(\vec{E}_\alpha^2/8\pi)$  from the maximum of the function  $\ln \Pi \equiv f(N_\alpha)$ ,

$$\ln \Pi = N(\ln N - 1) - \sum_{\alpha=1}^M N_\alpha (\ln N_\alpha - 1) \quad (7)$$

with

$$\sum_{\alpha=1}^M N_\alpha = N, \quad N \rightarrow \infty, \quad (8)$$

$$\sum_{\alpha=1}^M N_\alpha \vec{E}_\alpha^2/8\pi = N \frac{3}{2} (1 + Z^{-1}) nKT, \quad N \rightarrow \infty, \quad (9)$$

as constraints. Eq. (8) holds by definition of  $N$ , whereas Eq. (9) holds for a large number  $N$  of measurements and the average energy density  $\langle \vec{E}^2/8\pi \rangle$  of Eq. (2). Addition of the constraints (8) and (9) multiplied by the Lagrangian multipliers  $-\lambda$  and  $-\mu$  to Eq. (7) leads to the compact maximum conditions for  $\ln \Pi$ ,

$$\partial F(N_\alpha)/\partial N_\alpha = 0, \quad \partial^2 F(N_\alpha)/\partial N_\alpha^2 < 0, \quad \alpha = 1, 2, \dots, M, \quad (10)$$

where

$$F(N_\alpha) = N(\ln N - 1) - \sum_{\alpha=1}^M N_\alpha (\ln N_\alpha - 1) - \lambda \sum_{\alpha=1}^M N_\alpha - \mu \sum_{\alpha=1}^M N_\alpha \vec{E}_\alpha^2/8\pi. \quad (11)$$

The solution of Eq. (10) gives the distribution  $N_\alpha$  of the "discrete" energy densities  $\vec{E}_\alpha^2/8\pi$  in the form

$$N_\alpha = A e^{-\mu \vec{E}_\alpha^2/8\pi}, \quad A \equiv e^{-(1+\lambda)} \quad (12)$$

Henceforth, the subscript  $\alpha$  is dropped since  $\vec{E}_\alpha$  can be any point  $\vec{E}$  in the field space. The dimensional constants  $A(\lambda)$  and  $\mu$  are then given by the normalization conditions (8) and (9),

$$A \int_0^\infty e^{-\mu \vec{E}^2/8\pi} 4\pi \vec{E}^2 d\vec{E} = N, \quad (13)$$

$$A \int_0^{\infty} (\vec{E}^2/8\pi) e^{-\mu \vec{E}^2/8\pi} 4\pi E^2 dE = \frac{N^3}{2} (1+Z^{-1}) nKT, \quad (14)$$

as

$$A = [8\pi^2 (1+Z^{-1}) nKT]^{-3/2} N, \quad \mu = 1/(1+Z^{-1}) nKT. \quad (15)$$

For this normalization, which still contains the number  $N$  of measurements, the probability distribution (12) for the microfield energy density is

$$w_N(\vec{E}^2/8\pi) = \frac{N}{[8\pi^2 (1+Z^{-1}) nKT]^{3/2}} e^{-\vec{E}^2/8\pi (1+Z^{-1}) nKT}.$$

In theoretical applications, one is interested in the probability  $dP = w(\vec{E}) d^3\vec{E}$  for finding a microfield  $\vec{E}$  in the volume element  $d^3\vec{E}$  about the point  $\vec{E}$  of the field space, with the normalization  $\int dP = 1$ . The corresponding distribution function  $w(\vec{E})$  of the collective microfield  $\vec{E}$  is obtained by renormalization ( $N \rightarrow 1$ ):

$$w(\vec{E}) = [8\pi^2 (1+Z^{-1}) nKT]^{-3/2} e^{-\vec{E}^2/8\pi (1+Z^{-1}) nKT}. \quad (17)$$

This fundamental distribution has the form of a Gaussian, i.e. all its moments exist, e.g.,

$$\langle \vec{E}^0 \rangle = \iiint_{-\infty}^{+\infty} \vec{E}^0 w(\vec{E}) d^3\vec{E} = \vec{1}, \quad (18)$$

$$\langle \vec{E}^1 \rangle = \iiint_{-\infty}^{+\infty} \vec{E}^1 w(\vec{E}) d^3\vec{E} = \vec{0}, \quad (19)$$

$$\langle \vec{E}^2 \rangle = \iiint_{-\infty}^{+\infty} \vec{E}^2 w(\vec{E}) d^3\vec{E} = 12\pi (1+Z^{-1}) nKT. \quad (20)$$

The most probable ( $E_M$ ) and the r.m.s. ( $E_W$ ) collective microfields are by Eqs. (17) and (20)

$$E_M = [8\pi(1 + Z^{-1})nKT]^{1/2} \quad , \quad (21)$$

$$E_W = [12\pi(1 + Z^{-1})nKT]^{1/2} \quad . \quad (22)$$

For considerations concerning the fluctuation of the collective microfield  $\vec{E}(t)$  at a point  $\vec{r} \in \Omega$  with time  $t$ , temporal averages can be defined by

$$\overline{|\vec{E}|} = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{+\tau} |\vec{E}(t)| dt \quad , \quad (23)$$

$$\overline{\vec{E}^2} = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{+\tau} \vec{E}(t)^2 dt \quad . \quad (24)$$

The fluctuation of  $\vec{E}(t)$  is defined by  $\Delta\vec{E}(t) = \vec{E}(t) - \overline{\vec{E}(t)}$  with  $\overline{\vec{E}(t)} = \vec{0}$ . In stationary equilibrium, the time averages are identical with the ensemble averages.

By Eq. (17), the mean square (temporal) fluctuation of  $\vec{E}(t)$  is

$$\overline{\Delta E^2} = \overline{\vec{E}^2} - \overline{|\vec{E}|}^2 = (3 - \frac{8}{\pi})4\pi(1 + Z^{-1})nKT \quad . \quad (25)$$

TABLE I compares the r.m.s. field  $E_W$  and the r.m.s. fluctuation  $(\overline{\Delta E^2})^{1/2}$  of the collective microfield with the nearest neighbor Holtsmark field  $E_H = 2\pi^{1/3}(4/15)^{2/3} \times eZ^{1/3}n^{2/3}$  in dependence of the electron density  $n$  for typical ideal plasma conditions ( $\gamma \ll 1$ ,  $T = 10^4$ °K,  $Z = 1$ ). It is seen that  $E_W$  and  $(\overline{\Delta E^2})^{1/2}$  are one to two orders of magnitude larger than  $E_H$  in the range of ideal plasma densities  $n < 10^{18} \text{ cm}^{-3}$ . For these reasons, the Holtsmark field  $E_H$  represents a small contribution to the microfield in ideal plasmas. The result  $E_W \gg E_H$  is readily understandable since for ideal plasmas

$$E_H^2/E_W^2 = (\pi/3)(4/15)^{4/3}Z^{2/3}(1 + Z)^{-1} \cdot \frac{Ze^2 n^{1/3}}{KT} < \gamma \ll 1 \quad . \quad (26)$$

$n[\text{cm}^{-3}]$	$E_H[\text{Vcm}^{-1}]$	$E_W[\text{Vcm}^{-1}]$	$\overline{(\Delta E^2)^{1/2}}[\text{Vcm}^{-1}]$	$\gamma$
$10^{10}$	$1.741 \times 10^0$	$3.061 \times 10^2$	$1.190 \times 10^2$	$3.598 \times 10^{-4}$
$10^{12}$	$3.751 \times 10^1$	$3.061 \times 10^3$	$1.190 \times 10^3$	$1.670 \times 10^{-3}$
$10^{14}$	$8.081 \times 10^2$	$3.061 \times 10^4$	$1.190 \times 10^4$	$7.751 \times 10^{-3}$
$10^{16}$	$1.741 \times 10^4$	$3.061 \times 10^5$	$1.190 \times 10^5$	$3.598 \times 10^{-2}$
$10^{18}$	$3.751 \times 10^5$	$3.061 \times 10^6$	$1.190 \times 10^6$	$1.670 \times 10^{-1}$

TABLE I:  $E_H$ ,  $E_W$ ,  $\overline{(\Delta E^2)^{1/2}}$ , and  $\gamma$  versus  $n$  ( $T = 10^4 \text{°K}$ ,  $Z = 1$ ).

The probability for observing a collective microfield with intensity  $E = |\vec{E}|$  in the range between  $E$  and  $E + dE$  is  $P(E)dE = W(\vec{E})4\pi E^2 dE$ , where  $W(\vec{E})$  is given by Eq. (17). The maximum of the probability density  $P(E)$  is  $P(E_M) = 4e^{-1}[8\pi^2(1 + Z^{-1})nKT]^{1/2}$  by Eq. (21). Accordingly, the normalized probability density is  $P(E)/P(E_M) = (e/4)[8\pi^2(1 + Z^{-1})nKT]^{-1} \exp[-E^2/8\pi(1 + Z^{-1})nKT] \leq 1$ . Fig. 1 presents  $P(E)/P(E_M)$  versus  $0 \leq E \leq 10^8 [\text{Vcm}^{-1}]$  with  $nT = 10^{12} - 10^{22} [\text{cm}^{-3}\text{K}]$  as a parameter. This distribution is a displaced Gaussian with a maximum  $P(E)/P(E_M) = 1$  for  $E = E_m$ , which shifts to higher abscissas  $E_m = [8\pi \times (1 + Z^{-1})nKT]^{1/2}$  with increasing  $nT$ -values (pressures). The scattering width of the distributions  $\Delta E = \frac{1}{2}[8\pi(1 + Z^{-1})nKT]^{1/2}$  increases  $\propto (nT)^{1/2}$  with increasing  $nT$ -values (note logarithmic scale of abscissa). The increasing quantitative importance of the collective microfield in ideal plasmas with higher pressures  $p = (1 + Z^{-1})nKT$  is obvious.

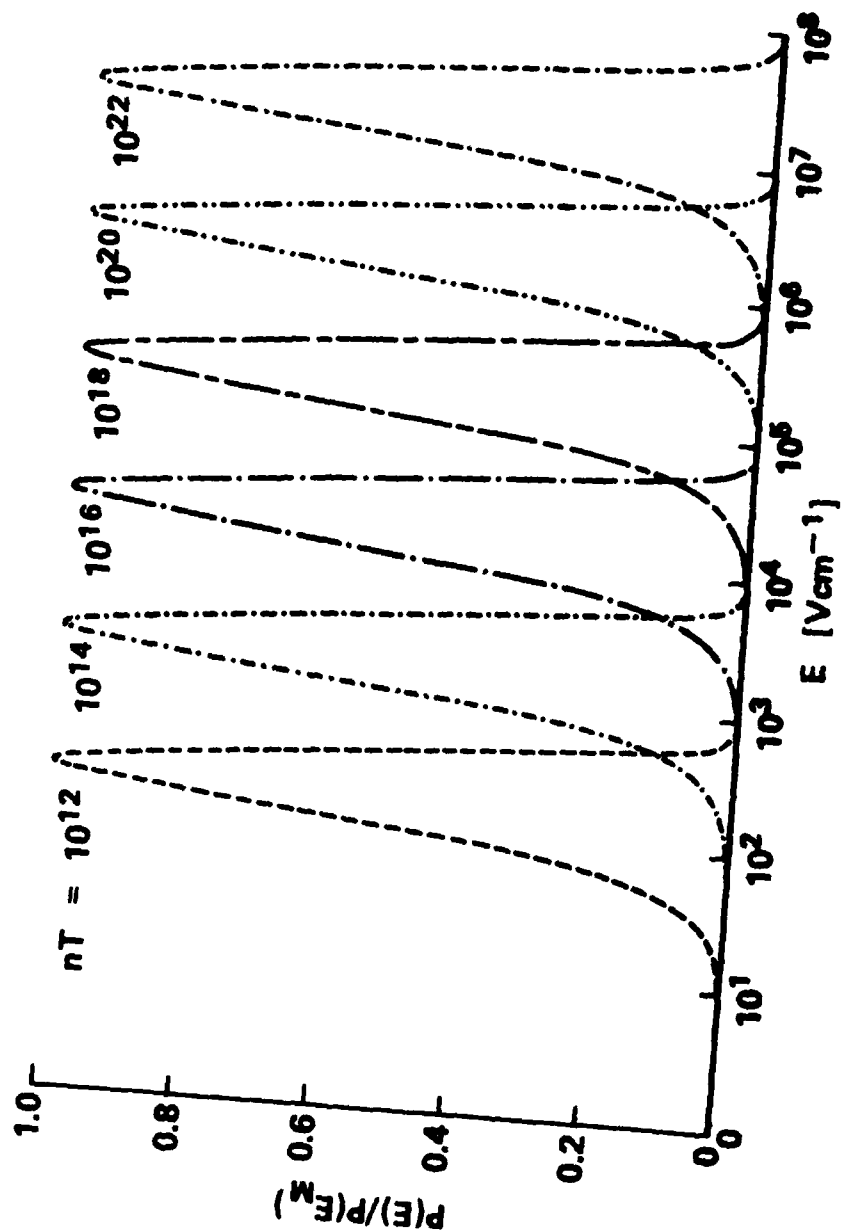


FIG.1: Normalized probability density  $P(E)/P(E_M)$  versus  $E$  for  $nT = 10^{12}$  to  $10^{22} \text{ cm}^{-3} \text{ } ^\circ\text{K}$ .

## ENERGY RELATIONS

A fully ionized plasma consisting of electrons and  $Z$  times ionized ions in thermal equilibrium at a temperature  $T$  exhibits various macroscopic energies, the average kinetic energy  $\langle K \rangle = \frac{3}{2}(1 + Z^{-1})nKT\Omega$ , the average electric field energy  $\langle U \rangle = \langle \vec{E}^2 / 8\pi \rangle \Omega$ , the average interaction energy  $\langle \phi \rangle$  and selfenergy  $\langle \psi \rangle$  of the electrons and ions. In order to derive the interrelation between these energies, the formation of the plasma by an electric charging process is considered. For this purpose, we assume that the electrons and ions are initially dispersed at infinity where they have only selfenergies. The plasma is then built up by moving one charge after the other from infinity into the volume  $\Omega$ , which requires work against the resulting Coulomb field of the charges already present in  $\Omega$ . The thermodynamics of the charging process is illustrated by i) a reversible isothermal and ii) an adiabatic or isotropic model.

The electric charging work expended in moving  $N$  electrons of charge  $e_e = -e$  and  $N/Z$  ions of charge  $Ze$  against their collective Coulomb field from infinity into the (finite) volume  $\Omega$  is (\* designates exclusion of terms with  $\mu = \nu$ )

$$A = \frac{1}{2} \sum_{\mu=1}^N \sum_{\nu=1}^N * e^2 \left| \frac{\vec{r}_\mu^e}{r_\mu^e} - \frac{\vec{r}_\nu^e}{r_\nu^e} \right|^{-1} + \frac{1}{2} \sum_{\nu=1}^{N/Z} \sum_{\mu=1}^{N/Z} * (Ze)^2 \left| \frac{\vec{r}_\nu^i}{r_\nu^i} - \frac{\vec{r}_\mu^i}{r_\mu^i} \right|^{-1} - \sum_{\mu=1}^N \sum_{\nu=1}^{N/Z} Ze^2 \left| \frac{\vec{r}_\mu^e}{r_\mu^e} - \frac{\vec{r}_\nu^i}{r_\nu^i} \right|^{-1} \quad (27)$$

where  $\vec{r}_\mu^{e,i}$  ( $\vec{r}_\nu^{e,i}$ ) are the position vectors of the  $\mu$ -th ( $\nu$ -th) electron (e) and ion (i) in the volume  $\Omega$ , respectively. The collective microfield of the  $N$  electrons and  $N/Z$  ions at a field point  $(\vec{r}, t)$  is the superposition

$$\vec{E}(\vec{r}, t) = \sum_{\mu=1}^N \vec{E}_\mu^e(\vec{r}, t) + \sum_{\nu=1}^{N/Z} \vec{E}_\nu^i(\vec{r}, t) \quad (28)$$

where  $\vec{E}_\mu^e(\vec{r}, t)$  and  $\vec{E}_\nu^i(\vec{r}, t)$  are the individual Coulomb fields produced at the field point  $(\vec{r}, t)$  by the  $\mu$ -th electron and the  $\nu$ -th ion, respectively. By Eq. (28), the electric field energy of the plasma  $\Omega$  is

$$U = \frac{1}{8\pi} \iiint_{\Omega} \vec{E}(\vec{r}, t)^2 d^3\vec{r} = \Phi + \Psi \quad (29)$$

where

$$\begin{aligned} \Phi = & \frac{1}{8\pi} \sum_{\mu=1}^{N^*} \sum_{\nu=1}^{N^*} \iiint_{\Omega} \vec{E}_\mu^e \cdot \vec{E}_\nu^e d^3\vec{r} + \frac{1}{8\pi} \sum_{\mu=1}^{N/Z^*} \sum_{\nu=1}^{N/Z^*} \iiint_{\Omega} \vec{E}_\mu^i \cdot \vec{E}_\nu^i d^3\vec{r} \\ & + \frac{1}{4\pi} \sum_{\mu=1}^N \sum_{\nu=1}^{N/Z} \iiint_{\Omega} \vec{E}_\mu^e \cdot \vec{E}_\nu^i d^3\vec{r} \end{aligned} \quad (30)$$

$$\Psi = \frac{1}{8\pi} \sum_{\mu=1}^N \iiint_{\Omega} (\vec{E}_\mu^e)^2 d^3\vec{r} + \frac{1}{8\pi} \sum_{\nu=1}^{N/Z} \iiint_{\Omega} (\vec{E}_\nu^i)^2 d^3\vec{r} \quad (31)$$

are the (e-e, i-i, and e-i) interaction energy and the (e and i) selfenergy of the plasma, respectively. Comparison of Eq. (27) with Eq. (29) reveals the interrelation

$$U - \Psi = \Phi = A. \quad (32)$$

Thus, we see that the field energy  $U$  is the sum of the interaction energy  $\Phi$  and the selfenergy  $\Psi$  [Eq. (29)]. The charging work  $A$  leads to an increase of the interaction part  $\Phi$  of the field energy  $U$  [Eq. (32)]. The selfenergy  $\Psi$  of the charges is independent of the spatial locations of the charges, i.e.  $\Psi$  is the same before and after the charging process.

Another independent energy relation is obtained by multiplication of the coupled Newtonian equations for the accelerations  $d^2\vec{r}_{\mu,\nu}^s(t)/dt^2$  of the  $\mu$ -th electron ( $s = e$ ) and the  $\nu$ -th ion ( $s = i$ ) by their respective velocities  $\vec{v}_{\mu,\nu}^s(\tau) = d\vec{r}_{\mu,\nu}^s(\tau)/dt$  and subsequent summation over all particles  $\mu$  and  $\nu$ . The resulting

expression can be brought into the form  $d(K + \Phi)/dt = 0$ , which demonstrates that the sum of kinetic ( $K$ ) and interaction ( $\Phi$ ) energies is an invariant  $H_0$ ,

$$K + \Phi = H_0 \quad (33)$$

where

$$K = \sum_{\mu=1}^N \frac{1}{2} m_e (\vec{v}_\mu^e)^2 + \sum_{\nu=1}^{N/Z} \frac{1}{2} m_i (\vec{v}_\nu^i)^2 \quad (34)$$

and  $\Phi=A$  is defined in Eq.(27). Eq. (33) expresses the conservation of kinetic  $K$  and interaction  $\Phi$  energies in a plasma, in which the electrons and ions interact by longitudinal Coulomb fields.

The thermodynamic functions of the plasma depend in general on the volume  $\Omega$ , the number  $N_s$  of particles in  $\Omega$ , and the particle averages of the random kinetic energies  $\frac{1}{2} m_s \vec{v}_s^2$  and the random field energy densities  $\vec{E}^2/8\pi$ . Accordingly, we assume  $U^{th} = U^{th}(T, \epsilon, N_s)$  for the thermal energy and  $S = S(T, \epsilon, N_s)$  for the entropy, where

$$3KT/2 = \langle \frac{1}{2} m_s \vec{v}_s^2 \rangle, \quad \epsilon = \langle \vec{E}^2/8\pi \rangle. \quad (35)$$

For plasma formation by isothermal reversible charging, the volume  $\Omega$  is embedded into a heat bath of temperature  $T$ . The transfer of  $dN_s$  charges  $e_s$  from infinity into the cavity  $\Omega$  requires on the average the charging work  $dA_s = d_s \langle U - \Psi \rangle = d_s \langle U \rangle$  by Eq. (32), and their thermalization at a temperature  $T$  consumes on the average the energy  $dU_s^{th} = \frac{3}{2} KT dN_s$  ( $s = e, i$ ). The difference of these energies,  $dQ_s$ , is supplied by the heat bath. Summation over "s" yields, in accordance with the first law of thermodynamics

$$dQ = dU^{th} - d\langle U \rangle \quad (36)$$

since no other than electric charging work is performed on the system ( $d\Omega = 0$ ).

The associated entropy  $dS = dQ/T$  is a complete differential,

$$dS = \frac{1}{T} \frac{\partial}{\partial T} (U^{th} - \langle U \rangle) dT + \frac{1}{T} \frac{\partial}{\partial \epsilon} (U^{th} - \langle U \rangle) d\epsilon + \frac{1}{T} \sum_{s=e,i} \frac{\partial}{\partial N_s} (U^{th} - \langle U \rangle) dN_s. \quad (37)$$

Application of the condition  $\partial_\epsilon \partial_T S = \partial_T \partial_\epsilon S$  to Eq. (37) yields the partial differential for constant  $N_s$  and  $T$ ,

$$\partial U^{th} / \partial \epsilon = \partial \langle U \rangle / \partial \epsilon. \quad (38)$$

Since  $U^{th} = 0$  for  $\epsilon = 0$  (no thermal energy in  $\Omega$  before charging), the integral of Eq. (38) is

$$U^{th} = \Omega \epsilon. \quad (39)$$

Eq. (39) could have been derived by other thermodynamic plasma formation processes, e.g. by adiabatic charging of the cavity  $\Omega$ . In this case  $dQ = 0$ , and by Eq. (36)

$$dQ = dU^{th} - d\langle U \rangle = 0 : \quad \langle U \rangle = U^{th}. \quad (40)$$

Finally,  $\langle U \rangle$  can also be determined as that equilibrium value which maximizes the entropy,

$$dS = T^{-1} [dU^{th} - d\langle U \rangle] = 0 : \quad \langle U \rangle = U^{th}. \quad (41)$$

Eqs. (39)-(41) indicate that an equipartition between thermal energy and average microfield energy exists in statistical equilibrium. This fundamental result is explicitly

$$\frac{3}{2} (1 + Z^{-1}) NKT = \Omega \langle E^2 / 8\pi \rangle. \quad (42)$$

The virial equation for the fully ionized plasma<sup>16)</sup>, the averages of the field energy equation (29), the energy conservation equation (33), and the energy balance equation (42) represent four independent equations for the average kinetic energy  $\langle K \rangle$ , the average field energy  $\langle U \rangle$ , the average interaction energy  $\langle \Phi \rangle$ , and the energy invariant  $H_0$ :

$$\langle K \rangle + \frac{1}{2} \langle \Phi \rangle = \frac{3}{2} p \Omega, \quad (43)$$

$$\langle U \rangle - \langle \Phi \rangle = \Psi, \quad (44)$$

$$\langle K \rangle + \langle \Phi \rangle = H_0, \quad (45)$$

$$\langle U \rangle = \langle K \rangle. \quad (46)$$

The pressure  $p$  of the plasma is assumed to be known (measurable). The self energy  $\Psi = \langle \Psi \rangle$  is independent of the random motion and spatial distribution of the particles in the case of ideal plasmas, and can be calculated from the charge distribution in the electrons and ions.

As an illustration, the plasma energies are calculated by means of Eqs. (43) - (46) for the case that  $p$  and  $\Psi$  are known:  $\langle U \rangle = \langle K \rangle = p\Omega + \Psi/3 > 0$ ,  $\langle \Phi \rangle = p\Omega - 2\Psi/3 < 0$ ,  $H_0 = 2p\Omega - \Psi/3 > 0$ . By definition, ideal plasmas are systems with weak Coulomb interactions, which preserve their electrical neutrality and collective behavior. In applications of Eqs. (43) - (46) to ideal plasmas, it should be kept in mind that the interaction energy  $\langle \Phi \rangle$  is always small but nevertheless nonvanishing,

$$0 < \gamma \sim |\langle \Phi \rangle| / \langle K \rangle \ll 1. \quad (47)$$

## CONCLUSION

In ideal plasmas, the distribution of the collective microfields is strongly temperature and density dependent. For typical temperatures and densities of ideal plasmas, the r.m.s. collective microfield is by orders of magnitude larger than the characteristic Holtsmark field. The temperature independent Holtsmark theory is approximately valid for strongly nonideal plasmas only, for which thermal effects are negligible. In statistical equilibrium, a balance among (average) kinetic particle and collective microfield energies exists, which is independent of the thermodynamic process of plasma formation.

#### REFERENCES

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1. J. Holtzmark, Ann. Physik 58, 577 (1919).
2. R. Gans, Ann. Physik 66, 396 (1921).
3. C. R. Vidal, Z. Naturforsch. 19a, 947 (1964).
4. A. A. Broyles, Phys. Rev. 100, 1181 (1955).
5. A. A. Broyles, Z. Phys. 151, 187 (1958).
6. M. Baranger and M. Mozer, Phys. Rev. 115, 521 (1959).
7. M. Baranger and M. Mozer, Phys. Rev. 118, 626 (1960).
8. G. Hettner and W. Wagner, Ann. Physik 4, 89 (1959).
9. G. Hettner and W. Wagner, Ann. Physik 5, 405 (1960).
10. K. Hunger and R. W. Larenz, Z. Physik 163, 245 (1961).
11. K. Hunger and R. W. Larenz, Z. Naturforsch. 23a, 1488 (1968).
12. H. E. Wilhelm, I.E.E.E. Trans. Plasma Science PS-8, 9 (1980).
13. R. C. Tolman, Principles of Statistical Mechanics (University Press, Oxford 1938).
14. H. E. Wilhelm, Ann. Physik 37, 241 (1980).
15. S. H. Kim and H. E. Wilhelm, Phys. Rev. A4, 2308 (1971); A5, 1813 (1972).
16. W. Schottky, Physik. Z. 21, 232 (1920).

## VII. FREE ENERGY OF NONIDEAL CLASSICAL AND DEGENERATE PLASMAS

By

H. E. Wilhelm and A. H. Khalfaoui

### Abstract

A quantum statistical theory of the free energy of a nonideal electron-ion plasma is developed for arbitrary interaction parameters  $0 < \gamma < \gamma_c$  ( $\gamma = Ze^2 n^{1/3}/KT$  is the ratio of mean Coulomb interaction and thermal energies), which takes into account the energy eigenvalues of (i) the thermal translational particle motions, (ii) the random collective electron and ion motions, and (iii) the static Coulomb interaction energy of the electrons and ions in their oscillatory equilibrium positions. From this physical model, the interaction part of the free energy is derived, which consists of a quasi-lattice energy depending on the interaction parameter  $\gamma$ , and the free energies of the quantized electron and ion oscillations (long range interactions). Depending on the degree of ordering, the Madelung "constant" of the plasma is  $\alpha(\gamma) = \bar{\alpha}$  for  $\gamma \gg 1$ ,  $\alpha(\gamma) \approx \bar{\alpha}$  for  $\gamma > 1$ , and  $\alpha(\gamma) \propto \gamma^{1/2}$  for  $\gamma \ll 1$ , where  $\bar{\alpha} \sim 1$  is a constant. The free energy of the high-frequency plasmons (electron oscillations) is shown to be very small for  $\gamma > 1$ , whereas the free energy of the low-frequency plasmons (ion oscillations) is shown to be significant for  $\gamma > 1$ , i.e. for proper nonideal conditions. For weakly nonideal plasmas,  $\gamma \ll 1$ , both the electron and ion oscillations contribute to the free energy. Thus, novel results are obtained not only for proper nonideal ( $\gamma > 1$ ) but also for weakly nonideal ( $\gamma \ll 1$ ) plasmas. From the general formula for the free interaction energy  $\Delta F$  of the plasma for  $0 < \gamma < \gamma_c$ , simple analytical expressions are derived for  $\Delta F$  in the limiting cases,  $\gamma \gg 1$ ,  $\gamma \approx 1$ , and  $\gamma \ll 1$ . Applications to astrophysical problems are discussed.

## INTRODUCTION

In the classical work of Debye and Hueckel on electrolytes, the total Coulomb interaction energy is calculated from the continuum theoretical picture of every ion interacting with its surrounding space charge cloud. Using more sophisticated methods, similar results were obtained for weakly nonideal plasmas ( $\gamma \ll 1$ ) by Mayer<sup>1</sup> (cluster expansion), Ichikawa<sup>2</sup> (collective variable approach<sup>3</sup>), Vedenov and Larkin<sup>4</sup> (graphical density expansion), and Jackson<sup>5</sup> (hydrodynamic continuum interaction model). Based on different methods and approximations, investigations of moderately ( $\gamma \gtrsim 1$ ) and strongly ( $\gamma \gg 1$ ) nonideal plasmas were given by Berlin and Montroll<sup>6</sup>, Theimer and Gentry<sup>7</sup>, Ecker and Kroell<sup>8</sup>, Ebeling, Hoffman and Kelbg<sup>9</sup>, and Varobev, Norman and Vilinov<sup>10</sup>, respectively.

In spite of differences in the theoretical approaches, the leading terms of the analytical results for proper nonideal plasmas ( $\gamma > 1$ ) give essentially the same formula for the free plasma energy,  $\Delta F/NKT = -a\gamma + b \ln \gamma + c$ , due to Coulomb interaction, where  $\gamma = Ze^2 n^{1/3}/KT$  is the ratio of electron - ion interaction energy and thermal energy, and  $a, b, c$  are constants depending on the respective approximations and assumptions. The thermodynamic functions of strongly nonideal plasmas ( $\gamma \gg 1$ ) were also determined with the help of Monte Carlo and computer methods by Brush, Sahlin, and Teller<sup>11</sup>, Hansen<sup>12</sup>, Vorobev et al<sup>13</sup> and Theimer<sup>14</sup>, respectively. Although computer methods provide limited physical insight, they are useful for checking the quantitative validity of analytical theories.

At sufficiently high electron densities, for which  $\gamma \gtrsim 1$ , classical statistical theories fail due to thermodynamic instability<sup>15</sup>, which is

inhibited by quantum mechanics. The classical plasma pressure would collapse for  $\gamma > 1$  due to the negative electron-ion interaction energy, whereas in reality the pressure remains positive in a plasma due to the Fermi pressure (exclusion principle) of the electrons. For these reasons, we present herein a quantum-statistical theory for nonideal plasmas based on concepts similar to those used by Debye for solids<sup>16)</sup>. The application of this model to proper nonideal plasmas ( $\gamma > 1$ ) is justified since a plasma exhibits a quasi-crystalline structure for  $\gamma > 0$  before it undergoes a diffuse transition into a solid, metallic state at a critical value  $\gamma_c$ . The role of the longitudinal phonons of the Debye theory is assumed by the quanta of the plasma oscillations (plasmons) in the case of the quasi-crystalline plasma. The theory is also applicable to weakly nonideal conditions, since the quasi-lattice energy reduces for weak ordering,  $\gamma \ll 1$ , to the free interaction energy of weakly nonideal plasmas.

The theory to be presented takes into consideration (i) the energy eigenvalues of the random, collective electron and ion oscillations and (ii) the static Coulomb interaction energy (quasi-lattice energy) of the electrons and ions in their oscillatory equilibrium positions. Thus, all significant long and short range Coulomb interactions are considered. The results are applicable to arbitrary nonideal plasmas,  $0 < \gamma < \gamma_c$ , where  $\gamma_c$  is the critical ordering parameter at which a phase transition into a solid metallic state occurs.

## PHYSICAL FOUNDATIONS

Subject of the theoretical considerations are quasi-homogeneous high-pressure plasmas consisting of electrons of charge  $-e$  and density  $n = N/V$  and ions of charge  $+Ze$  and density  $n/Z = N/ZV$ , with typical densities in the range  $10^{20} \text{ cm}^{-3} \leq n \leq 10^{24} \text{ cm}^{-3}$  and temperatures of the order-of-magnitude  $T \sim 10^3 - 10^4 \text{ }^\circ\text{K}$ . For these conditions, the Debye radius  $D = [4\pi n e^2 (1+Z)/KT]^{-1/2}$  is  $D = 6.901 \times [T/n(1+Z)]^{1/2} \leq 10^{-8} \text{ cm}$ , i.e.,  $D$  is smaller than the atomic dimension and the number of particles in the Debye sphere would be  $N_D = 4\pi n D^3/3 \leq 1$  for  $D < 10^{-8} \text{ cm}$  and  $n < 10^{24} \text{ cm}^{-3}$ . It is seen that the concept of Debye shielding completely breaks down, and statistical theories containing the Debye length as a characteristic parameter would be physically meaningless for high density plasmas.

The nonideal behavior of plasmas is determined by the interaction parameter  $\gamma$ , which is the ratio of the Coulomb interaction energy  $-Ze^2 n^{1/3}$  and thermal energy  $-KT$ ,

$$\gamma = Ze^2 n^{1/3}/KT = 1.671 \times 10^{-3} Z n^{1/3}/T. \quad (1)$$

It follows that  $0.5Z \leq \gamma \leq 15Z$  for  $10^{20} \text{ cm}^{-3} \leq n \leq 10^{24} \text{ cm}^{-3}$  and  $T \sim 10^4 \text{ }^\circ\text{K}$ . For  $\gamma \approx 1$ , the nature of the plasma changes from a "thermally expanding" ( $\gamma < 1$ ) to an "electrostatically contracting" ( $\gamma > 1$ ) plasma. For  $\gamma > 1$ , the collapse of the plasma due to Coulomb attraction between electrons and ions is inhibited by the Fermi pressure of the electrons, i.e. by the quantum mechanical exclusion principle. Thus, in the region  $0 < \gamma < \gamma_c$  the plasma undergoes a diffuse transition from a nonideal classical plasma ( $\gamma < 1$ ) to a quasi-crystalline plasma ( $1 \leq \gamma < \gamma_c$ ), with an incomplete ordering comparable to that of a liquid.

An understanding of strongly nonideal plasmas has been attempted via the model of discrete interacting particles in a dense gas<sup>(6-14)</sup>. For the above reasons, however, it appears to be more adequate to calculate

the thermodynamic functions of proper nonideal plasmas from the picture of collective electron and ion oscillations. In this approach, the free interaction energy is due to the static Coulomb interaction of the electrons and ions in their "equilibrium positions" (Madelung energy) and their oscillation energies about the equilibrium positions (plasmon energies).

Since the plasma volume  $V$  contains  $N$  electrons and  $N/Z$  ions, there exist  $3N$  (high-frequency branch) and  $3N/Z$  (low frequency branch) characteristic frequencies  $\omega_i$  of longitudinal oscillations. Each plasma oscillator of frequency  $\omega_i$  can only have the energy  $(n_i + \frac{1}{2})\hbar\omega_i, n_i=0,1,2,\dots$ , so that the energy  $E\{i\}$  of a plasma state with  $n_i$  plasmons of frequency  $\omega_i$  is

$$E\{i\} = \sum_{\{i\}} n_i \hbar \omega_i \quad (2)$$

where  $\{i\}$  designates the entire set of given eigenfrequencies  $\omega_i$ . Accordingly, the partition function  $Q$  of the longitudinal plasma oscillations is

$$Q = \sum_{\{i\}} e^{-E\{i\}/KT} = \prod_{\{i\}} 1/(1 - e^{-\hbar\omega_i/KT}). \quad (3)$$

From  $Q$ , the thermodynamic functions such as the pressure, internal energy, entropy, etc., are derived in the usual way, e.g., the free energy of the plasmons is

$$\tilde{F} = -KT \ln Q = KT \sum_{\{i\}} \ln(1 - e^{-\hbar\omega_i/KT}). \quad (4)$$

In the limit  $V \rightarrow \infty$ , the discrete eigenfrequencies  $\omega_i$  are replaced by continuous ones,  $\omega = \omega(k)$ , in accordance with the dispersion law for space charge waves of wave length  $\lambda = 2\pi/k$ ,  $0 \leq k \leq \hat{k}$ .

1. Electron Oscillations. The high-frequency branch of the space charge waves is due to longitudinal electron oscillations. Their frequency  $\omega$  is for classical ( $n \ll \tilde{n}$ ) and completely degenerate ( $n \gg \tilde{n}$ ) electrons given by<sup>17)</sup>

$$\omega^2 = \omega_p^2 [1 + (\kappa_e/4\pi) Z \gamma^{-1} (k r_e)^2] \quad , \quad n \ll \tilde{n} \quad , \quad (5)$$

$$\omega^2 = \omega_p^2 \left[ 1 + \frac{9}{20\pi} \left( \frac{\pi}{6} \right)^{1/3} \left( \frac{n}{\tilde{n}} \right)^{2/3} (Z/\gamma) (k \bar{r}_e)^2 \right], \quad n \gg \tilde{n}, \quad (6)$$

where

$$\tilde{n} = 2(2\pi m k T / h^2)^{3/2}, \quad (7)$$

$$\omega_p = (4\pi n e^2 / m)^{1/2}, \quad (8)$$

$$\bar{r}_e = n^{-1/3}, \quad (9)$$

are the critical electron density, the plasma frequency, and the mean electron distance ( $\kappa_e \leq c_p / c_v$  of the electrons, and  $m$  is their mass). Since  $k_{\max} \sim 2\pi / \bar{r}_e$  (oscillations with  $\lambda < \bar{r}_e$  are physically inconceivable), the electron oscillations propagate,  $\omega = \omega(k) > \omega_p$ , in nonideal plasmas.

2. Ion Oscillations. The low-frequency branch of the space charge waves is essentially due to ion sound waves. Since the ions are presumed to be nondegenerate, the frequency of the ion oscillations is given by<sup>17)</sup>

$$\omega = \delta(k) (\kappa_i k T / M)^{1/2} k \quad (10)$$

where

$$\delta(k) = \left[ 1 + \frac{Z(\kappa_e / \kappa_i)}{1 + (\kappa_e / 4\pi) Z \gamma^{-1} (k \bar{r}_e)^2} \right]^{1/2}, \quad n \ll \tilde{n}, \quad (11)$$

$$\delta(k) \approx 1, \quad n \gg \tilde{n}, \quad (12)$$

is a correction factor of magnitude-of-order 1, which shows the influence of the electrons on the ion oscillations ( $M = \text{mass}$ ,  $\kappa_i = c_p / c_v$  of the ions).

In weakly nonideal plasmas,  $\gamma \ll 1$ , the electron sound waves are strongly damped for wave lengths  $\lambda < D$ , due to trapping of the resonance

electrons with thermal speeds comparable to the wave speed. For proper nonideal plasmas,  $\gamma > 1$ , the number of particles in the Debye sphere  $4\pi D^3/3$  is no longer large compared with one and  $D < 10^{-8}$  cm is smaller than atomic size, so that thermal Landau damping is no longer feasible. For this reason, electron oscillations should exist for wave length  $\lambda \geq \bar{r}_e$  if  $\bar{r}_e > D$ .

The ions are nondegenerate since  $n_i \ll g_i (2\pi MKT/h^2)^{3/2}$  for the  $n - T$  region under consideration. The electrons are considerable degenerate for  $n > \tilde{n}$  by Eq. (7), i.e. their kinetic energy is essentially given by the Fermi energy  $E_F = \hbar^2 (3\pi^2 n)^{2/3} / 2m$  for  $n > \tilde{n}$ . For this reason, the nonideality of the electrons increases with increasing  $n$  as long as  $n < \tilde{n}$ , but then decreases with increasing  $n$  as soon as  $n > \tilde{n}$ . From the condition  $Ze^2 n^{1/3} = E_F$  follows that the electrons form again an ideal gas for  $n \gg 10^{23} Z^3$ . This anomalous behavior is explained by the stronger increase of  $E_F \propto n^{2/3}$  with  $n$  compared with the Coulomb energy  $E_c \propto n^{1/3}$ .

It is recognized that the effects of degeneracy and nonideality on the dispersion of the ion sound waves, Eq. (10), are negligible. Similarly, the effect of nonideality on the dispersion of the sound waves of the degenerate electrons, Eq. (6), is negligible, but in the dispersion equation of the classical electrons, Eq. (5),  $\kappa_e$  has to be interpreted as a polytropic coefficient, where  $\kappa_e(\gamma) \sim c_p/c_v$  as to order-of-magnitude.

# STATISTICAL THERMODYNAMICS

In the plasma under consideration, the electrons and ions interact through their longitudinal Coulomb fields (transverse electromagnetic interactions are negligible for  $KT \ll mc^2$ ). The electrons ( $s = e$ ) and ions ( $s = i$ ) have thermal velocities  $\vec{c}_s$  and random collective mean mass velocities  $\vec{v}_s$  due to their oscillatory wave motions about the equilibrium positions, so that their local velocity is  $\vec{v}_s = \vec{v}_s + \vec{c}_s$ , with  $\langle \vec{c}_s \rangle = 0$  and  $\langle \vec{v}_s \rangle = \vec{v}_s$ , where  $\langle \vec{u}_s \rangle = \iiint \vec{u}_s f_s d^3 \vec{v}_s$  is the average of  $\vec{u}_s$  with respect to the normalized velocity distribution  $f_s$  of the species  $s$ . The resulting Hamilton function with Coulomb interaction leads to a free energy of the plasma of the form:

$$F = \sum_{s=e,i} F_s^{(0)} + E_M + \sum_{s=e,i} \tilde{F}_s \quad (13)$$

$F_s^{(0)}$  is the ideal free energy of the noninteracting plasma components  $s$ .  $E_M$  is the Coulomb interaction energy of the electrons and ions in their equilibrium positions.  $\tilde{F}_{e,i}$  is the free energy of the electron and ion oscillations, i.e. of the high and low frequency plasmons, Eq. (4).

It should be noted that Eq. (9) takes into consideration all significant short-range and long-range Coulomb interactions by means of the Madelung energy  $E_M$  and the plasmon energies  $\tilde{F}_s$ . As is evident from the derivation<sup>17)</sup> of Eqs. (5)-(6) and (10), in which terms of order  $m/M$  are neglected compared to 1, Eq. (9) contains the e-e, e-i, and i-i Coulomb interactions at distances  $\lambda > n^{-1/3}$ .

1. Free Energy  $F_s^{(0)}$ . In high pressure plasmas, the electrons are partially degenerate for densities  $n > \tilde{n}$  where  $\tilde{n} = 4.828 \times 10^{15} T^{3/2} [\text{cm}^{-3}]$ , whereas the ions behave in general classically. Fermi statistics gives for the free energy of the ideal electron gas<sup>18)</sup>

$$F_e^{(0)} = - NKT U_{3/2}(\mu/KT)/U_{1/2}(\mu/KT) \quad (14)$$

where

$$U_\rho(\mu/KT) = \frac{1}{\Gamma(\rho+1)} \int_0^\infty \frac{x^\rho dx}{e^{x-\mu/KT} + 1}, \quad \rho=1/2, 3/2 \quad (15)$$

and

$$n = 2(2\pi mKT/h^2)^{3/2} U_{1/2}(\mu/KT) \quad (16)$$

defines the Sommerfeld integrals <sup>19)</sup>, and determines the chemical potential  $\mu = \mu(n, T)$  of the electrons, respectively. The free energy of the translational degrees of freedom of the classical, ideal ion gas is <sup>18)</sup>

$$F_i^{(0)} = - (N/Z) KT \ln[(2\pi mKT/h^2)^{3/2} \left(\frac{Z}{n}\right)]. \quad (17)$$

2. Quasi-Lattice Energy  $E_M$ . The equilibrium positions of the electrons and ions, about which the electrostatic oscillations occur, form an electron "lattice" and an ion "lattice", with an incomplete ordering. In the Wigner-Seitz approximation, the Coulomb interaction energy of the electron-ion lattices is, independent of the lattice type,

$$E_M = - \alpha \gamma NKT, \quad \alpha \approx \bar{\alpha} = \frac{9}{10} (4\pi/3Z)^{1/3}, \quad \gamma > 1. \quad (18)$$

As the ordering of the plasma increases with  $\gamma$ ,  $\alpha(\gamma)$  is a weak function of  $\gamma$  such that asymptotically  $\alpha = \bar{\alpha}$  for  $\gamma \gg 1$ . Eq.(18) indicates that  $-E_M/N \sim Ze^2/\bar{r}_i$  is of the order of the average e-i interaction energy. For weak ordering,  $\gamma \ll 1$ , it will be shown that  $\alpha \propto \gamma^{1/2}$ .

3. High-Frequency Contribution  $\tilde{F}_e$ . Since the number of longitudinal modes with wave numbers between  $k$  and  $k + dk$  in volume  $V$  is  $V 4\pi k^2 dk / (2\pi)^3$ , Eq. (4) gives for the free energy  $\tilde{F}_e$  of the high-frequency electron oscillations of energy  $\hbar\omega(k)$

$$\tilde{F}_e / KT(V/2\pi^2) = \int_0^{\hat{k}} e^{\ln\{1 - \exp[-\hbar\omega(k)/KT]\}} k^2 dk \quad (19)$$

where

$$\omega(k) = \omega_p (1 + a^2 k^2)^{1/2}, \quad (20)$$

$$a^2 \equiv c_m^2 / \omega_p^2 = (\kappa_e / 4\pi) (Z/\gamma) \bar{r}_e^2, \quad n \ll \tilde{n}, \quad (21)$$

$$a^2 \equiv (3/5) v_F^2 / \omega_p^2 = \frac{9}{20\pi} \left(\frac{\pi}{6}\right)^{1/3} \left(\frac{n}{\tilde{n}}\right)^{2/3} (Z/\gamma) \bar{r}_e^2, \quad n \gg \tilde{n}, \quad (22)$$

by Eqs. (5)-(6). The speed of sound  $c_m$  and the Fermi speed  $v_F$  of the electrons are

$$c_m = (\kappa_e KT/m)^{1/2}, \quad v_F = \hbar(3\pi^2 n)^{1/3}/m. \quad (23)$$

The number of modes in  $(0, \hat{k}_e)$  and  $V$  equals the number  $3N$  of degrees of freedom of the electron gas, i.e.,

$$(2\pi)^{-3} V \int_0^{\hat{k}_e} 4\pi k^2 dk = 3N, \quad \hat{k}_e = (18\pi^2 n)^{1/3}. \quad (24)$$

Integration of Eq. (19) by parts yields, under consideration of  $\hat{k}_e^3 KT / 6\pi^2 = 3NKT$ , for the free energy of the high-frequency plasmons:

$$\tilde{F}_e = 3NKT \left( \ln\{1 - \exp[-\frac{\hbar\omega_p}{KT} (1 + a^2 \hat{k}_e^2)^{1/2}]\} - F\left(\frac{\hbar\omega_p}{KT}, a\hat{k}_e\right) \right) \quad (25)$$

where

$$F\left(\frac{\hbar\omega_p}{KT}, a\hat{k}_e\right) = \frac{\hbar\omega_p}{KT} (a\hat{k}_e)^{-3} \int_0^{\hat{k}_e} \frac{x^4 (1+x^2)^{-1/2} dx}{e^{(\hbar\omega_p/KT)(1+x^2)^{1/2}} - 1} \quad (26)$$

and

$$\hbar\omega_p / KT = (4\pi)^{1/2} (\lambda_e / \bar{r}_e) (\gamma/Z)^{1/2}, \quad \lambda_e = \hbar / (mKT)^{1/2}, \quad (27)$$

$$\hat{a}k_e = \kappa_e^{1/2} (9\pi^{1/2}/4)^{1/3} (Z/\gamma)^{1/2}, \quad n \ll \tilde{n}, \quad (28)$$

$$\hat{a}k_e = 2^{1/6} \pi^{1/3} (9/2\sqrt{5}) (n/\tilde{n})^{1/3} (Z/\gamma)^{1/2}, \quad n \gg \tilde{n}. \quad (29)$$

By means of the successive substitutions, (i)  $x = \sinh \xi$ ,  $dx = \cosh \xi d\xi$  and (ii)  $\epsilon = (\hbar\omega_p/KT) \cosh \xi$ ,  $d\epsilon = (\hbar\omega_p/KT) \sinh \xi d\xi$ , the integral (26) is transformed to

$$F(\epsilon_p, \hat{a}k_e) = (\hat{a}k_e \epsilon_p)^{-3} \int_{\epsilon_p}^{\hat{\epsilon}_e} (\epsilon^2 - \epsilon_p^2)^{3/2} (e^\epsilon - 1)^{-1} d\epsilon \quad (30)$$

where

$$\epsilon_p = \hbar\omega_p/KT, \quad \hat{\epsilon}_e = \epsilon_p [1 + (\hat{a}k_e)^2]^{1/2}. \quad (31)$$

Since the leading expression in Eq. (25) is the logarithmic term, it is sufficient to give for  $F(\epsilon_p, \hat{a}k_e)$  the series approximation (Appendix),

$$F(\epsilon_p, \hat{a}k_e) / 2^{3/2} (\hat{a}k_e)^3 \epsilon_p^{-3/2} = \sum_{m=1}^{\infty} e^{-m\epsilon_p} \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)_n (2\epsilon_p)^{-n} m^{-(\frac{5}{2}+n)} \gamma(\frac{5}{2}+n, (\hat{\epsilon}_e - \epsilon_p)m), \quad \hat{\epsilon}_e < 3\epsilon_p, \quad (32)$$

where

$$\gamma(\frac{5}{2}+n, (\hat{\epsilon}_e - \epsilon_p)m) = m^{\frac{5}{2}+n} \int_0^{\hat{\epsilon}_e - \epsilon_p} u^{\frac{3}{2}+n} e^{-mu} du \quad (33)$$

is the incomplete gamma function<sup>20)</sup>. Since in general  $\gamma/Z \gtrsim 1$  for  $\epsilon_p < \hat{\epsilon}_e < 3\epsilon_p$ , the expansion (32) is useful where simple approximate relations do not exist.

4. Low Frequency Contribution  $\tilde{F}_1$ . With the number of modes in the interval  $dk$  at  $k$  and volume  $V$  given by  $V4\pi k^2 dk / (2\pi)^3$ , Eq. (4) yields for the free energy  $\tilde{F}_1$  of the low-frequency ion oscillations of energy  $\hbar\omega(k)$

$$\tilde{F}_1 / KT(V/2\pi^2) = \int_0^{\hat{k}_1} \ln \{1 - \exp[-\hbar\omega(k)/KT]\} k^2 dk \quad (34)$$

where

$$\omega(k) = \delta(k) c_M k, \quad (35)$$

$$c_M = (\kappa_1 KT/M)^{1/2} \quad (36)$$

by Eqs. (10) and (12). The number of modes in  $(0, \hat{k}_1)$  and  $V$  equals the number  $3N/Z$  of degrees of freedom of the ion gas, i.e.,

$$(2\pi)^{-3} V \int_0^{\hat{k}_1} 4\pi k^2 dk = 3N/Z, \quad \hat{k}_1 = (18\pi^2 n/Z)^{1/3} \quad (37)$$

Partial integration of Eq. (34) gives, under consideration of

$\hat{k}_1^3 KT / 6\pi^2 = 3(N/Z)KT$ , for the free energy of the low-frequency plasmons:

$$\tilde{F}_1 = 3(N/Z)KT \left( \ln \{1 - \exp[-\frac{\hbar c_M}{KT} \delta(\hat{k}_1) \hat{k}_1]\} - G(\hat{k}_1) \right) \quad (38)$$

where

$$G(\hat{k}_1) = \frac{\hbar c_M}{KT} \hat{k}_1^{-3} \int_0^{\hat{k}_1} \frac{[\delta(k) + k\delta'(k)] k^3 dk}{e^{(\hbar c_M/KT)\delta(k)k} - 1} \quad (39)$$

Since the dispersion factor  $\delta(k)$  is a bounded function varying very little with  $k$  such that  $1 \leq \delta(k) \leq (1+Z)^{1/2}$  for  $k \in (0, \hat{k}_1)$ ,  $\delta(k)$  can be approximated by an average value  $\bar{\delta}$ ,

$$\delta(k) = \bar{\delta} \sim 1, \quad n \lesssim \bar{n} \quad (40)$$

Since in addition the logarithmic expression is the dominant term in Eq. (38),

the integral (39) is approximated by

$$G(\hat{k}_1) \approx \hat{k}_1^{-3} \int_0^{\hat{k}_1} \epsilon^3 (e^\epsilon - 1)^{-1} d\epsilon \quad (41)$$

where

$$\epsilon = \hbar c_M \bar{\delta} k / KT, \quad \hat{\epsilon}_1 = \hbar c_M \bar{\delta} \hat{k}_1 / KT. \quad (42)$$

$G(\hat{\epsilon}_1)$  has the semi-convergent series expansions,<sup>20)</sup>

$$G(\hat{\epsilon}_1) = \frac{1}{3} \left[ 1 - \frac{3}{8} \hat{\epsilon}_1 + \frac{1}{20} \hat{\epsilon}_1^2 + \dots \right], \quad \hat{\epsilon}_1 \ll 1, \quad (43)$$

$$G(\hat{\epsilon}_1) = \frac{\pi^4}{15} \hat{\epsilon}_1^{-3} + O[e^{-\hat{\epsilon}_1}], \quad \hat{\epsilon}_1 \gg 1. \quad (44)$$

This completes the formal mathematical aspects of the theory,  
the physical implications of which require further elaboration.

## APPLICATIONS

For applications of the theory to strongly, intermediate, and weakly nonideal plasmas, it should be noted that the dimensionless parameters  $\gamma/Z$ ,  $\hbar\omega_p/KT$ ,  $\hat{a}k_e$ , and  $n/\tilde{n}$  occurring in Eq. (25) for the free energy  $\tilde{F}_e$  of the high-frequency plasmons can not be varied independently. Since both  $\gamma/Z$  and  $\lambda_e/\bar{r}_e$  increase with increasing  $n$  and decreasing  $T$ ,  $\hbar\omega_p/KT \sim (\lambda_e/\bar{r}_e)(\gamma/Z)^{1/2}$  varies over a large  $n$ - $T$  region similar to  $(\gamma/Z)^{1/2}$ , Eq. (7). Numerically,

$$\begin{aligned}\gamma/Z &= 1.670 \times 10^{-3} n^{1/3}/T, \quad \hbar\omega_p/KT = 4.328 \times 10^{-7} n^{1/2}/T, \quad n/\tilde{n} = 2.071 \times 10^{-16} n T^{-3/2}, \\ \hat{a}k_e &= 1.586 \kappa_e^{1/2} (\gamma/Z)^{-1/2}, \quad n \ll \tilde{n} \\ \hat{a}k_e &= 3.308 (n/\tilde{n})^{1/3} (\gamma/Z)^{-1/2}, \quad n \gg \tilde{n}.\end{aligned}\quad (45)$$

E. g., for  $T=10^4$  K,  $\gamma/Z \gtrsim 1$  if  $n \gtrsim 10^{21} \text{ cm}^{-3}$  and  $\hbar\omega_p/KT \gtrsim 1$  if  $n \gtrsim 5 \times 10^{20} \text{ cm}^{-3}$ . For  $T=10^3$  K,  $\gamma/Z \gtrsim 1$  if  $n \gtrsim 10^{18} \text{ cm}^{-3}$ , etc. Thus, for typical conditions of nonideal plasmas  $\gamma/Z$  and  $\hbar\omega_p/KT$  are of the same order of magnitude. It is also recognized that in general  $n/\tilde{n} \gg 1$  if  $\gamma/Z \gg 1$ , and  $n/\tilde{n} \ll 1$  if  $\gamma/Z \ll 1$ .

In Eq. (38) for the free energy  $\tilde{F}_i$  of the low frequency plasmons, only one characteristic parameter  $\hat{\epsilon}_i$  occurs since  $\delta(k) \sim \bar{\delta} \sim 1$ . By Eq. (42), this parameter is

$$\hat{\epsilon}_i = \frac{\hbar c_M \bar{\delta} k_i}{KT} = (18\pi^2)^{1/3} \kappa_i^{1/2} \bar{\delta} \frac{\lambda_i}{\bar{r}_i} = 2.158 \times 10^{-5} Z^{-1/3} \left(\frac{m}{M}\right)^{1/2} \frac{\pi^{1/3}}{T^{1/2}} \bar{\delta} \ll 1 \quad (46)$$

where

$$\lambda_i = \hbar/(MKT)^{1/2}, \quad \bar{r}_i = (n/Z)^{-1/3}. \quad (47)$$

Accordingly, for typical nonideal plasma conditions, it is  $\hat{\epsilon}_i \ll 1$  since  $\lambda_i/\bar{r}_i \ll 1$  (classical ions) although in general  $\lambda_e/\bar{r}_e > 1$  (degenerate electrons) for  $\gamma/Z > 1$  or  $\hbar\omega_p/KT > 1$ .

The deviation  $\Delta F$  of the free energy of a nonideal plasma from ideality is by Eq. (13) due to the quasi-lattice energy  $E_M$  and the plasmon energies  $\tilde{F}_{e,i}$ ,

$$\Delta F = E_M + \sum_{s=e,i} \tilde{F}_s. \quad (48)$$

Since the theory of electron oscillations<sup>17)</sup> has not yet been developed for arbitrary degrees of degeneracy ( $n \lesssim \tilde{n}$ ), the contributions of the electron oscillations to  $\Delta F$  in the cases  $n \lesssim \tilde{n}$  and  $n \gtrsim \tilde{n}$  have to be estimated from the dispersion equations for  $n \ll \tilde{n}$  [Eq. (5)] and  $n \gg \tilde{n}$  [Eq. (6)], respectively. Fortunately, it turns out that  $|\tilde{F}_e| \ll |\Delta F|$  for  $\gamma/Z \gtrsim 1$ , so that quantitatively reliable approximations for  $\Delta F$  can be derived.

1. Strongly Nonideal Plasmas. By Eq. (6) the spectrum  $\omega(k)$  of electron oscillations extends over a band  $\Delta\omega \sim \omega_p$  above the plasma frequency for  $\gamma/Z \gg 1$  since  $k\tilde{r}_e \leq \hat{k}_e \tilde{r} \sim 1$  and  $(n/\tilde{n})^{2/3} Z\gamma^{-1} \sim 1$ . Application of the mean value theorem for integrals to Eq. (25) shows that the free energy  $\tilde{F}_e$  of the high-frequency plasmons vanishes exponentially for  $\epsilon_p \rightarrow \infty$ , i.e.  $\gamma/Z \rightarrow \infty$ :

$$\begin{aligned} \tilde{F}_e/3NKT &= \left( \ln\{1 - \exp[-\epsilon_p (1+a^2 \hat{k}_e^2)^{1/2}]\} \right. \\ &\quad \left. - \frac{\epsilon_p (a\hat{k}_e)^{-3}}{\exp[\epsilon_p (1+\tilde{x}^2)^{1/2}] - 1} \int_0^{a\hat{k}_e} x^4 (1+x^2)^{-1/2} dx \right) \rightarrow 0, \quad \epsilon_p \rightarrow \infty; \\ 0 &\leq \tilde{x} \leq a\hat{k}_e. \end{aligned} \quad (49)$$

Accordingly,  $|\tilde{F}_e|/3NKT \ll 1$  for  $\epsilon_p \gg 1$ , i.e.,  $\gamma/Z \gg 1$ . On the other hand, the free energy of the low frequency plasmons is by Eq. (38) for nondegenerate ions

$$\tilde{F}_1 \approx 3(N/Z)KT[\ln \hat{\epsilon}_1 - (1/3)] = 3(N/Z)KT\{\ln \gamma + \ln[(18\pi^2/Z^4)^{1/3} \frac{(\kappa_1 KT/M)^{1/2}}{e^2/\hbar}] - (1/3)\}, \quad \hat{\epsilon}_1 \ll 1. \quad (50)$$

It is noted that  $\gamma/Z \gg 1$  is compatible with  $\hat{\epsilon}_1 = \hbar c_M \hat{k}_1 \bar{\delta}/KT \ll 1$  as explained above.

Equations (49) and (50) demonstrate that the contribution of the electron oscillations to the free energy is negligible in strongly nonideal plasmas,  $\gamma/Z \gg 1$ . In this limit, the nonideal part of the free energy is due to the quasi-lattice energy  $E_M$  and the ion oscillations,

$$\Delta F/NKT = -\bar{\alpha}\gamma + (3/Z)\ln \gamma + (3/Z)\ln(\beta c_M/v_B) - (1/Z), \quad \gamma/Z \gg 1, \quad (51)$$

where

$$v_B = e^2/\hbar, \quad \beta = (18\pi^2 Z^{-4})^{1/3}. \quad (52)$$

Note that  $\ln \gamma$  depends on both  $n$  and  $T$  whereas  $\ln \beta c_M/v_B$  depends only on  $T$ , where the Bohr speed is  $v_B = 2.118 \times 10^8$  cm/sec  $\gg c_M = (\kappa_1 KT/M)^{1/2}$ .

It is remarkable that the electron oscillations contribute little to the free energy compared to the ion oscillations for  $\gamma/Z \gg 1$ . This result holds even for moderately nonideal conditions,  $\gamma/Z > 1$ . Thus, we disagree with the formula " $F = n\epsilon_0 + 3NKT \ln(\hbar\omega_0/KT)$ " stated without derivation for nonideal plasmas by Norman and Starostin<sup>21</sup>, according to whom "all the vibrations have exactly the same frequency  $\omega_0$  near the plasma frequency  $\omega_p$ ". The derivation of this formula requires  $\hbar\omega(k)/KT \ll 1$  for the electron oscillations, which implies  $\gamma/Z \ll 1$ , but the latter inequality contradicts their assumption  $\omega(k) \approx \omega_0 \approx \omega_p$ , since the frequency spectrum extends over a large band  $\Delta\omega > \omega_p$  above  $\omega_p$  for  $\gamma/Z \ll 1$ . For these reasons, the free energy proposed by them is not applicable to proper nonideal plasmas,  $\gamma/Z > 1$ , nor is it correct for less nonideal conditions,  $\gamma/Z < 1$ .

2. Intermediate Nonideal Plasmas. For intermediate nonideal conditions,  $1 \leq \gamma/Z < 10$ , the spectrum  $\omega(k)$  of electron oscillations extends over a region  $\Delta\omega < 0[\omega_p]$  above  $\omega_p$  by Eq. (6) since  $(n/\tilde{n})^{2/3}Z\gamma^{-1} < 1$  and  $k\tilde{r}_e \leq \hat{k}_e \tilde{r}_e \sim 1$ . Also in this case, a relatively simple formula can be devised for the free energy. The logarithmic term in  $\tilde{F}_e$ , Eq. (25) is negligible compared to that in  $\tilde{F}_i$ , Eq. (38), for  $\gamma/Z > 1$  since  $\epsilon_p \gg \hbar c_M \hat{\delta} k_1 / KT$  for  $\gamma/Z > 1$  by Eqs. (45) and (46), respectively. Accordingly, the nonideal part (48) of the free energy is for intermediate nonideal plasmas:

$$\begin{aligned} \Delta F/NKT = & -\bar{\alpha}\gamma + (3/Z)\ln\gamma + (3/Z)\ln(\beta c_M/v_B) - (3/Z)G(\hat{\epsilon}_1) \\ & - 3F(\epsilon_p, ak_e), \quad \gamma/Z \geq 1. \end{aligned} \quad (53)$$

For  $\gamma/Z \geq 1$ , the ions can be assumed to be non-degenerate,  $\hat{\epsilon}_i = \pi c_M \bar{\delta} k_i / KT \ll 1$  by Eq. (46), so that the ion integral (41) reduces to

$$G(\hat{\epsilon}_i) = 1/3, \quad \hat{\epsilon}_i \ll 1. \quad (54)$$

Since  $\hat{\epsilon}_e > \epsilon_p \geq 1$  and  $ak_e \epsilon_p \geq 1$  [Eq. (45)] for  $\gamma/Z \geq 1$ , the electron integral (30) is significantly smaller than  $G(\hat{\epsilon}_i) = 1/3$ ,

$$0 < F(\epsilon_p, ak_e) < (\hat{\epsilon}_e^2 - \epsilon_p^2)^{3/2} (ak_e \epsilon_p)^{-3} \ln(1 - e^{-\hat{\epsilon}_e} / 1 - e^{-\epsilon_p}) \ll 1, \quad \gamma/Z \geq 1. \quad (55)$$

The lower and upper bounds of  $F(\epsilon_p, ak_e)$  have been obtained by means of the mean value theorem for the integral (30),

$$F(\epsilon_p, ak_e) = (ak_e \epsilon_p)^{-3} (\hat{\epsilon}_e^2 - \epsilon_p^2)^{3/2} \int_{\epsilon_p}^{\hat{\epsilon}_e} (e^{\epsilon} - 1) d\epsilon, \quad \epsilon_p \leq \hat{\epsilon} \leq \hat{\epsilon}_e. \quad (56)$$

While for strongly nonideal conditions, the contribution of the electron oscillations to the free energy is completely negligible, this contribution is still insignificant for intermediate nonideal conditions,  $\gamma/Z \geq 1$ , by Eq. (55). For more exact evaluations, the small term  $F(\epsilon_p, ak_e)$  in Eq. (53) can be computed from Eq. (30) or (32).

3. Weakly Nonideal Plasmas. Although the theory of weakly nonideal systems is well understood,<sup>1-5)</sup> it is interesting to investigate whether the present model for proper nonideal plasmas gives reasonable results in the limit  $\gamma/Z \ll 1$ . For  $\gamma/Z \ll 1$  it is  $ak_e \gg 1$  by Eq. (45), and the spectrum  $\omega(k)$  of electron oscillations extends over a large region  $\Delta\omega \gg \omega_p$  above  $\omega_p$  by Eq. (5). The electron integral becomes for  $ak_e \gg 1$ ,

$$F(\epsilon_p, ak_e) = \epsilon_p (ak_e)^{-3} \int_0^{ak_e} (e^{\epsilon_p x} - 1)^{-1} x^3 dx, \quad \gamma/Z \ll 1, \quad (57)$$

i.e.,

$$F(\epsilon_p, ak_e) = 1/3 \left[ 1 - \frac{3}{8} (\epsilon_p ak_e)^{-1} + \frac{1}{20} (\epsilon_p ak_e)^{-2} - \dots \right], \quad \epsilon_p ak_e \ll 1. \quad (58)$$

Although  $\epsilon_p \hat{a} k_e$  is independent of  $\gamma/Z$  by Eqs. (27) and (28), the expansion (58) is valid since the electrons are certainly nondegenerate,  $\lambda_e/\bar{r} \ll 1$  for  $\gamma/Z \ll 1$ , and

$$\epsilon_p \hat{a} k_e = (4\pi\kappa_e)^{1/2} (9\pi^{1/2}/4)^{1/3} \lambda_e/\bar{r}_e \ll 1, \quad \lambda_e/\bar{r}_e \ll 1. \quad (59)$$

For nondegenerate ions, the integral (41) is  $G(\hat{\epsilon}_i) = 1/3$  by Eq. (43) since  $\hat{\epsilon}_i \ll 1$ . Thus, one obtains from Eqs. (18), (25) and (38) for the interaction part of the free energy of weakly nonideal plasmas:

$$\begin{aligned} \Delta F/NKT = & -\alpha(\gamma)\gamma + (3/Z)\ell n\gamma + (3/Z)\ell n(\beta c_M/v_B) \\ & + 3\ell n(\epsilon_p \hat{a} k_e) - (1+Z^{-1}), \quad \gamma/Z \ll 1, \end{aligned} \quad (60)$$

where the logarithmic term in Eq. (25) has been expanded for

$$\epsilon_p \hat{a} k_e \ll 1.$$

In Eq. (60),  $\alpha(\gamma)$  is the Madelung constant of the weakly nonideal plasma with weak electron and ion ordering,  $\alpha(\gamma) \rightarrow 0$  for  $\gamma \rightarrow 0$ . Comparison of the term  $-\alpha(\gamma)\gamma(NKT)$  in Eq. (60) with  $\Delta F =$

$$-(NKT)(2/3)\pi^{1/2}(1+Z)^{3/2}e^{3n^{1/2}}(KT)^{-3/2} \text{ of the Debye-Hueckel theory}^{23)}$$

(weakly nonideal plasmas) yields the result

$$\alpha(\gamma) = (2/3)\pi^{1/2} (1+Z^{-1})^{3/2} \gamma^{1/2}, \quad \gamma/Z \ll 1. \quad (61)$$

The previous theories of weakly nonideal plasmas do not lead to the logarithmic terms in Eq. (60) since they do not take into account the effects of electron and ion oscillations.

4. Numerical Illustrations. Fig. 1 shows the (negative) free energy  $F_0$  of an ideal  $Z = 1$  plasma versus  $n$  and  $T$  based on Eqs. (14) - (17).  $F_0$  serves as a reference quantity, relative to which the quantitative significance of the nonideal contributions are measured.  $|F_0|$  increases with increasing  $n$  and  $T$ .

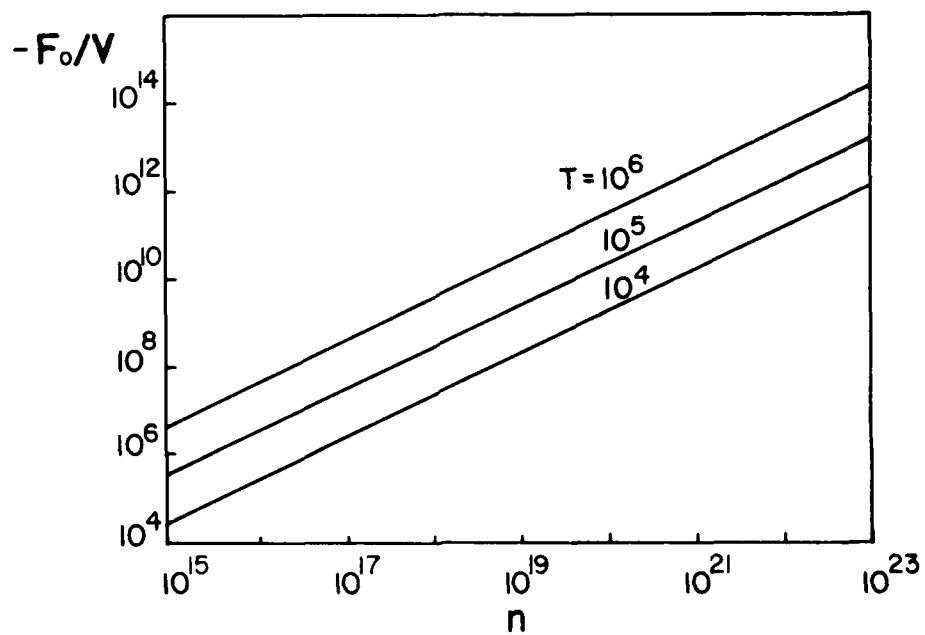


FIG.1: Free energy  $F_0 < 0$  of ideal plasma versus  $n[\text{cm}^{-3}]$  with  $T[^\circ\text{K}]$  as parameter ( $Z=1$ ).

Fig. 2 shows the deviation  $\Delta F < 0$  of the free energy of a  $Z = 1$  plasma from its ideal value  $F_0 < 0$  versus  $n$  and  $T$  based on Eqs. (48), (25) and (38). In the  $n$ - $T$  region under consideration,  $|\Delta F|$  is of the same magnitude-of-order as  $|F_0|$ , i.e. is considerably larger than the thermal energy  $-NKT$ .  $\Delta F/F_0$  exhibits only at large densities  $n > 10^{19} \text{ cm}^{-3}$  a significant  $T$  - dependence.

Fig. 3 shows the free energies  $\tilde{F}_e$  and  $\tilde{F}_i$  of the high (e) and low (i) frequency plasmons of a  $Z = 1$  plasma based on Eqs. (25) and (38).  $|\tilde{F}_i|$  is considerable larger than  $|\tilde{F}_e|$ , in particular at higher densities. The  $T$ -dependence of  $\tilde{F}_{e,i}/F_0$  increases with increasing density  $n$ . Comparison of Figs. 2 and 3 indicates that  $\Delta F \approx \tilde{F}_e + \tilde{F}_i$ , i.e. the quasi-lattice energy  $E_M$  [Eqs. (18) and (61)] is not the dominant nonideal effect.

The Figs. 2 and 3 demonstrate the quantitative importance of the nonideal effects  $\Delta F = E_M + \tilde{F}_e + \tilde{F}_i$ , in particular of the low (i) and high (e) frequency plasmon contributions  $\tilde{F}_i$  and  $\tilde{F}_e$  ( $\tilde{F}_i > \tilde{F}_e$ ), for the evaluation of the free energy  $F = F_0 + \Delta F$  of high density plasmas.

For quantitative calculations, it is noted that the free energy  $\Delta F$  is hardly affected by inaccuracies in the large maximum wave numbers  $\hat{k}_e$  and  $\hat{k}_i$ , which have been determined in accordance with the Debye theory which implies strong coupling ( $\gamma \gg 1$ ). For weakly nonideal plasmas,  $\gamma \ll 1$ , it appears to be more meaningful to determine  $k_s = 2\pi/\hat{\lambda}_s$  from the minimum wave length  $\hat{\lambda}_s \approx 2r_s$  where  $r_s = (4\pi n_s/3)^{-1/3}$  is the mean particle radius,  $s = e, i$ . Both models give, however, essentially the same result since  $\hat{k}_s^D / \hat{k}_s^\lambda = (3/\pi) \cdot (\pi/2)^{1/3} \approx 1$ .

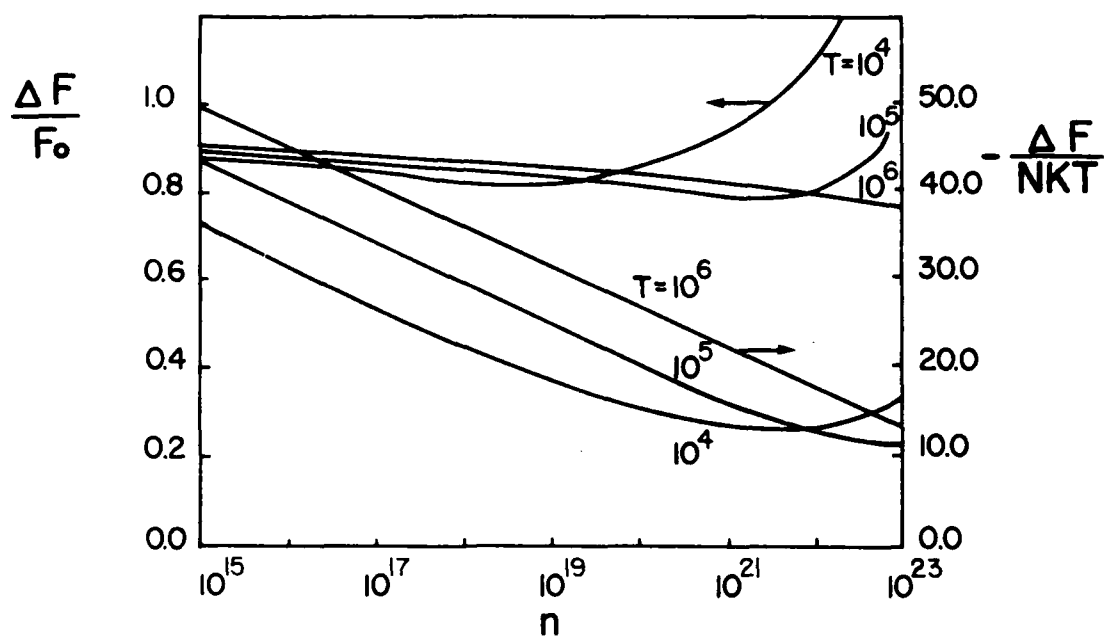


FIG.2: Deviation  $\Delta F < 0$  of free energy from  $F_0 < 0$  versus  $n[\text{cm}^{-3}]$  with  $T[^\circ\text{K}]$  as parameter ( $Z=1$ ).

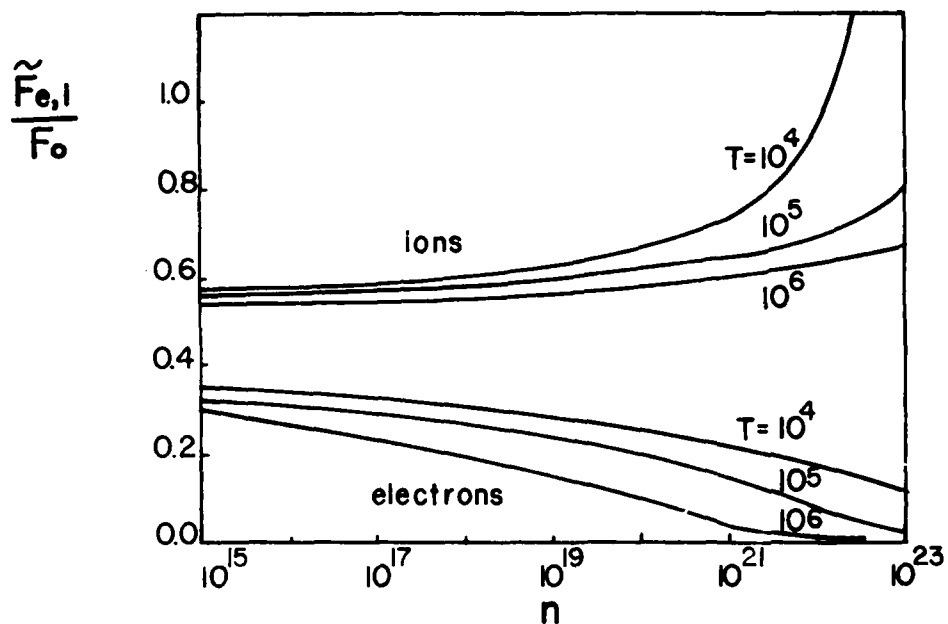


FIG.3: Free energies  $\tilde{F}_{e,i} < 0$  of high (e) and low (i) frequency plasmons versus  $n[\text{cm}^{-3}]$  with  $T[^\circ\text{K}]$  as parameter ( $Z=1$ ).

# APPENDIX: Expansion of $F(\epsilon_p, \hat{a}k_e)$

The integral (23) is conveniently rewritten in the form

$$F(\epsilon_p, \hat{a}k_e) = (\hat{a}k_e \epsilon_p)^{-3} I(\epsilon_p, \hat{\epsilon}) \quad (A1)$$

where

$$I(\epsilon_p, \hat{\epsilon}) = \int_{\epsilon_p}^{\hat{\epsilon}} (\epsilon^2 - \epsilon_p^2)^{3/2} (e^\epsilon - 1)^{-1} d\epsilon, \quad 0 < \epsilon_p < \hat{\epsilon} < \infty. \quad (A2)$$

Since  $\epsilon > 0$ , i.e.  $e^{-\epsilon} < 1$ , there exists the series expansion,

$$(e^\epsilon - 1)^{-1} = \sum_{m=1}^{\infty} e^{-m\epsilon}, \quad \epsilon > 0. \quad (A3)$$

The substitution,  $u = \epsilon - \epsilon_p$ ,  $du = d\epsilon$ , and Eq. (A3) transform Eq. (A2) to

$$I(\epsilon_p, \hat{\epsilon}) = \sum_{m=1}^{\infty} e^{-m\epsilon_p} \int_{u=0}^{\hat{\epsilon}-\epsilon_p} u^{3/2} (u+2\epsilon_p)^{3/2} e^{-mu} du. \quad (A4)$$

For  $u < 2\epsilon_p$ , i.e.,  $\hat{\epsilon} < 3\epsilon_p$ , the binomial expansion,

$$(u+2\epsilon_p)^{3/2} = (2\epsilon_p)^{3/2} \sum_{n=0}^{\infty} \binom{3/2}{n} \left(\frac{u}{2\epsilon_p}\right)^n, \quad u/2\epsilon_p < 1, \quad (A5)$$

is used, which reduces Eq. (A4) to the double series:

$$I(\epsilon_p, \hat{\epsilon}) = (2\epsilon_p)^{3/2} \sum_{m=1}^{\infty} e^{-m\epsilon_p} \sum_{n=0}^{\infty} \binom{3/2}{n} (2\epsilon_p)^{-n} m^{-(\frac{5}{2}+n)} \gamma\left(\frac{5}{2}+n, (\hat{\epsilon}-\epsilon_p)m\right), \quad \hat{\epsilon} < 3\epsilon_p, \quad (A6)$$

where

$$\gamma\left(\frac{5}{2}+n, (\hat{\epsilon}-\epsilon_p)m\right) = m^{\frac{5}{2}+n} \int_0^{\hat{\epsilon}-\epsilon_p} u^{\frac{3}{2}+n} e^{-mu} du. \quad (A7)$$

is the incomplete gamma function, which is tabulated.<sup>20)</sup> In an analogous way, the integral (A2) can be solved for  $u > 2\epsilon_p$ , i.e.,  $3\epsilon_p < \hat{\epsilon} < \infty$ .

#### REFERENCES

1. J. E. Mayer, J. Chem. Phys. 18, 1426 (1950).
2. Y. H. Ichikawa, Progr. Theoret. Phys. 20, 715 (1958).
3. D. Bohm and D. Pines, Phys. Rev. 82, 625 (1951); 92, 609 (1953).
4. A. A. Vedenov and A. I. Larkin, Sov. Phys. JETP 8, 806 (1959).
5. J. L. Jackson and L. S. Klein, Physics of Fluids 7, 232 (1964).
6. T. Berlin and E. Montroll, J. Chem. Phys. 20, 75 (1952).
7. O. Theimer and R. Gentry, Ann. Phys. 17, 93 (1962).
8. G. Ecker and W. Kroell, Phys. Fluids 6, 62 (1963).
9. W. Ebeling, H. J. Hoffmann, and G. Kelbg, Beitr. Plasma Physik 8, 233 (1967).
10. V. S. Vorobev, G. E. Norman, and V. S. Filinov, Sov. Phys. JETP 30, 459 (1970).
11. S. Brush, H. L. Sæhlin, and E. Teller, J. Chem. Phys. 45, 2102 (1966).
12. J. P. Hansen, Phys. Rev. A8, 3096 (1973).
13. V. S. Vorobev, G. E. Norman, and V. S. Filinov, Sov. Phys. JETP 28, 838 (1969).
14. O. Theimer and M. M. Theimer, Am. J. Phys. 46, 661 (1978).
15. G. E. Norman and A. N. Starostin, High Temperature 6, 410 (1968).
16. P. Debye, Ann. Physik 39, 789 (1912).
17. A. G. Sitenko, Electromagnetic Fluctuations in Plasma (Academic, New York 1967).
18. R. C. Tolman, Principles of Statistical Mechanics (Oxford University Press, Oxford, 1938).
19. A. Sommerfeld, Z. Physik 47, 1 (1928).
20. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover Publ., New York, 1965).
21. G. E. Norman and A. N. Starostin, High Temperature 8, 413 (1970).
22. Ya. B. Zeldovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, I (Academic Press, New York 1966).
23. P. Debye and E. Hueckel, Phys. Z. 24, 185 (1923).

## VIII. FREE ENERGY OF RANDOM SOUND OSCILLATIONS

By

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### ABSTRACT

Thermal equilibrium properties of a monatomic gas are investigated by taking into account the energies of the random sound wave oscillations. The free energy is derived by a quantum statistical mechanics due to Bose. The system is considered as a macroscopic continuum in which random acoustic oscillations are thermally excited. It is shown that the contribution  $\Delta F$  to the free energy due to the sound waves is significant for high density gases, in particular at moderately high temperatures.

## INTRODUCTION

In solids and liquids, the effect of sound waves on the thermodynamic quantities was studied by Landau and others<sup>(1)</sup>. In ionized gases, the electrical oscillations (plasma oscillations) affect the thermal equilibrium of the system<sup>(2-3)</sup>. Similarly, we are considering the contribution of the sound wave oscillations to the free energy of noncondensed gases. In thermal equilibrium of gases, the acoustic oscillations share the partition of energy and, thus, change the thermodynamic functions of the system. The distribution of the sound wave quanta is determined by Bose statistics, which is used herein.

The problem under consideration is concerned with gases as a macroscopic continuum, which exhibits a set of separate elementary excitations, the sound wave oscillations. These excitations behave like "quasi-particles" moving in the volume occupied by the gas, and have definite energies. The free energy of the gas evaluated by the theory to be presented will take into consideration, in addition to the random thermal energies of the gas particles, the energy of the random sound wave oscillations. It will be shown, that the effect of the sound waves is important only at high temperatures and high gas densities. The results of this theory are applicable at temperatures and densities for which the gas is not in a condensed state (liquid or solid).

Although nonideal effects due to finite particle size are not taken into account explicitly, it should be noted that the gas under consideration is not an ideal one. The existence of sound waves in the gas implies that there are particle interactions, since a gas can not perform the ordered, collective mean mass motions of random sound waves without such interactions.

## THEORY

Consider a gas as a continuum of volume  $V$  containing  $N$  atoms. Because the velocity of the gas in a sound wave is in the direction of propagation, the sound waves are longitudinal. Each oscillator of frequency  $\omega_\sigma$  of the longitudinal sound waves can only have the energies ( $\hbar = h/2\pi =$  reduced Planck constant)

$$E_{n_\sigma} = \hbar \omega_\sigma (n_\sigma + \frac{1}{2}), \quad n_\sigma = 0, 1, 2, \dots \infty. \quad (1)$$

The frequency of the sound waves with wave number  $k_\sigma$  is ( $M =$  mass of atoms)

$$\omega_\sigma = k_\sigma C_s, \quad C_s = (\gamma KT/M)^{1/2} \quad (2)$$

where  $\gamma = C_p/C_v$  is the polytropic coefficient. Accordingly, the partition function of the gas oscillations is:

$$Z = \prod_\sigma \sum_{n_\sigma=0}^{\infty} e^{-\beta \hbar C_s k_\sigma (n_\sigma + \frac{1}{2})} = \prod_\sigma \frac{e^{-\beta \hbar C_s k_\sigma / 2}}{1 - e^{-\beta \hbar C_s k_\sigma}} \quad (3)$$

where  $\beta = 1/KT$  ( $K =$  Boltzmann constant,  $T =$  Temperature of the system). From the partition function  $Z$ , the thermodynamic quantities, such as pressure, internal energy etc., are derived in the usual way. The free energy of the random sound oscillations is given by:

$$\Delta F = -KT \ln Z. \quad (4)$$

In the limit  $V \rightarrow \infty$ , the discrete eigenfrequencies  $\omega_\sigma$  are replaced by a continuous spectrum,  $\omega = \omega(k)$ , in accordance with the dispersion law for sound waves, of wave length  $\lambda = 2\pi/k$ ,

$$\omega = k C_s, \quad 0 \leq k \leq \hat{k}. \quad (5)$$

The theory to be presented is sensitive towards the cut-off wave number  $\hat{k}$ , which is large in all cases of interest. Since acoustic waves with wavelengths  $\lambda < \max(\bar{r}, L)$  and mean free paths  $L < \bar{r} = n^{-1/3}$  are not possible in gases,

$k = 2\pi/\lambda$  is determined by the mean free path  $L$ ,

$$\hat{k} = \frac{2\pi}{L}, \quad L = L(n, T), \quad (6)$$

where  $n = N/V$  is the density of the atoms. The number of (longitudinal) wave modes with wave numbers between  $k$  and  $k + dk$  in volume  $V$  is  $g(k)dk = V 4\pi k^2 dk / (2\pi)^3$ .

Accordingly, Eqs. (3) and (4) give

$$\Delta F = - \frac{KTV}{2\pi^2} \int_0^{\hat{k}} k^2 \ln \left( \frac{e^{-\beta \hbar C_s k/2}}{1 - e^{-\beta \hbar C_s k}} \right) dk. \quad (7)$$

The integral in Eq. (7) is decomposed into the contributions i) from the ground state ( $n_0 = 0$ ) and ii) higher states ( $n_0 > 0$ ). By Eqs. (3) and (7)

$$\Delta F = F_1 + F_2 \quad (8)$$

where

$$F_1 = \frac{V \hbar C_s}{4\pi^2} \int_0^{\hat{k}} k^3 dk = V \hbar C_s \left( \frac{\hat{k}^2}{4\pi} \right)^2 \quad (9)$$

and

$$F_2 = \frac{V \hat{k}^3}{6\beta\pi^2} \ln(1 - e^{-\beta \hbar C_s \hat{k}}) - \frac{V \hbar C_s}{6\pi^2} \int_0^{\hat{k}} \frac{k^3 dk}{e^{\beta \hbar C_s k} - 1} \quad (10)$$

by partial integration. The integral in Eq. (10) can be solved for "high" and "low" temperatures by series expansions which give<sup>(4,5)</sup>:

$$F_2 = \frac{V \hat{k}^3}{6\beta\pi^2} \left\{ \ln(1 - e^{-x}) - \frac{1}{3} + \frac{x}{8} + x \sum_{v=1}^{\infty} \frac{B_{2v} x^{2v}}{(2v+3)2v!} \right\}, \quad x < 2\pi \quad (11)$$

and

$$F_2 = \frac{V \hat{k}^3}{6\beta\pi^2} \left\{ \ln(1 - e^{-x}) - \frac{1}{x^3} \left[ 6\zeta(4) - \sum_{n=1}^{\infty} e^{-nx} \left( \frac{1}{nx} + \frac{3}{n^2 x^2} + \frac{6}{n^3 x^3} + \frac{6}{n^4 x^4} \right) \right] \right\},$$

$$x \geq 1, \quad (12)$$

where

$$x = \beta \hbar C_s \hat{k} \quad (13)$$

and  $B_{2\nu}$  are the Bernoulli numbers, and  $\zeta(4) = \pi^4/90$  is the Riemann  $\zeta$ -function.

For comparison purposes,  $\Delta F$  and the classical free energy  $F_0$  of the ideal monatomic gas ( $M$  = atomic mass) are stated:

$$F_0 = NKT \left[ \ln \left\{ \left( \frac{h^2}{2\pi MKT} \right)^{3/2} n \right\} - 1 \right] \quad (14)$$

and

$$\Delta F = V\kappa C_s \left( \frac{\hat{k}^2}{4\pi} \right)^2 + \frac{V\hat{k}^3}{6\beta\pi^2} \left\{ \ln(1 - e^{-x}) - \frac{1}{3} + \frac{x}{8} - \sum_{\nu=1}^{\infty} \frac{B_{2\nu} x^{2\nu}}{(2\nu+3) 2\nu!} \right\}, \quad x < 2\pi. \quad (15)$$

$$\Delta F = V\kappa C_s \left( \frac{\hat{k}^2}{4\pi} \right)^2 + \frac{V\hat{k}^3}{6\beta\pi^2} \left\{ \ln(1 - e^{-x}) - \frac{1}{x^3} \left[ 6\zeta(4) - x^4 \sum_{n=1}^{\infty} e^{-nx} \right] \left( \frac{1}{nx} + \frac{3}{n^2 x^2} + \frac{6}{n^3 x^3} + \frac{6}{n^4 x^4} \right) \right\}, \quad x \geq 1, \quad (16)$$

for "high" and "low" temperatures, respectively.

It is interesting to compare the specific heat of the ideal gas ( $C_0$ ) and the sound oscillations ( $\Delta C$ ) in the high temperature limit,  $x = \beta\kappa C_s \hat{k} \ll 1$ .

By Eqs. (14) and (15),

$$C_0 = -T \partial^2 F_0 / \partial T^2 = 3NK/2 \quad (17)$$

$$\Delta C = -T \partial^2 \Delta F / \partial T^2 = \frac{2\pi}{3} (V/L^3) K \ll C_0 \quad \text{for } \bar{r} = n^{-1/3} \ll L. \quad (18)$$

In the derivation of Eq. (18), it should be noted that  $\Delta F \approx (V\hat{k}^3/6\beta\pi^2) \ln x$  for  $x \ll 1$  where  $x \propto T^{-1/2}$ . It is seen that  $\Delta C < C_0$  at high temperatures since  $\bar{r} < L$ .

This completes the mathematical aspects of the problem, the physical implications of which will be discussed next.

## DISCUSSION

In a hypothetical ideal equilibrium gas without particle interactions, the average particle velocity is  $\langle \vec{c} \rangle = \int \vec{c} f(\vec{c}) d^3\vec{c} = \vec{0}$ , i.e. the particles have pure thermal velocities  $\vec{c}$  with a Maxwell distribution  $f(\vec{c})$ . No random mean mass motions or collective particle motions, such as sound oscillations, exist due to the absence of particle interactions. The free energy of the ideal or non-interacting monatomic gas is, therefore,  $F_0$ , Eq.(14). In any real gas with particle interactions, stochastic mean mass motions  $\langle \vec{v}(\vec{r}, t) \rangle = \int \vec{v} f(\vec{v}, \vec{r}, t) d^3\vec{v} \neq \vec{0}$  exist due to the presence of thermally excited sound waves [ $f(\vec{v}, \vec{r}, t)$  is the local distribution of actual particle velocities  $\vec{v} = \langle \vec{v} \rangle + \vec{c}$ ]. Since the total energy of the gas is distributed both over the thermal particle motion  $\vec{c}$  and the stochastic, acoustic mean mass motions  $\langle \vec{v}(\vec{r}, t) \rangle$ , a free energy contribution from the random sound waves exists. Thus, the free energy  $\Delta F$  of the random sound oscillations represents a nonideal effect which is ultimately due to particle interactions, which make a hydrodynamic or continuum description of a gas possible.

In the contribution  $\Delta F$  of the sound waves to the free energy  $F$  of an ideal gas as given by Eq.(5) or (16), we identify two parts. i)  $F_1$  which is the contribution of the "zero oscillation" mode which corresponds to  $n_0 = 0$  in Eq.(1), and ii)  $F_2$  the higher mode oscillation contributions  $n_0 > 1$ . The explanation for the increase of the free energy of the sound quanta  $\hbar\omega$  with temperature is given by statistics. In the high temperature limit, the number  $N_\omega$  of sound quanta of frequency  $\omega$  is  $N_\omega \approx KT/\hbar\omega$ , i.e. increases proportional with  $T$ .

In Fig. 1 we have drawn  $\Delta F/F_0$  for monatomic helium gases over a range of temperatures and densities to show the variation of  $\Delta F$ . The overall contribution of  $\Delta F$  is larger at higher densities but decreases rapidly for lower densities, especially at high temperatures. Quantitatively,  $\Delta F$  represents a noticeable effect only at extremely high densities  $n$  of the gas [ $n > 10^{21} \text{ cm}^{-3}$ ]. For this reason, the free energy  $\Delta F$  of the sound oscillations has to be considered in the evaluation of the thermodynamic functions of high-temperature gases only at high densities.

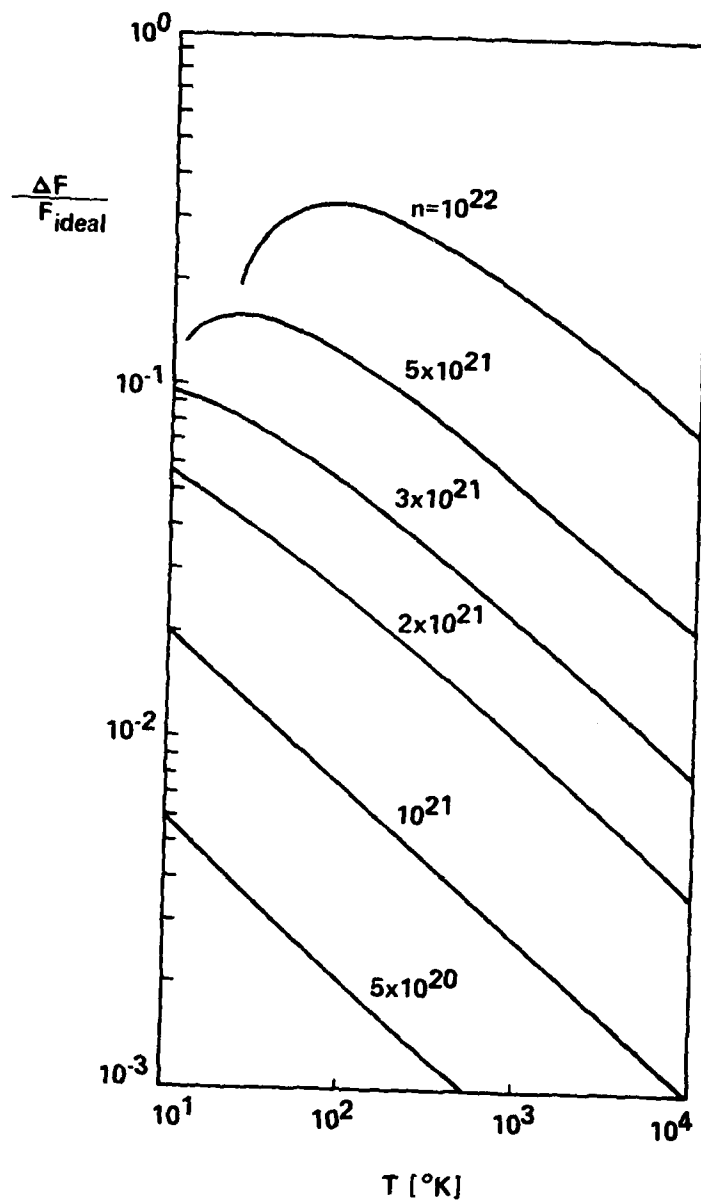


Fig. 1: Free energy  $\Delta F$  due to sound waves in helium gas as function of temperature  $T [^{\circ}K]$  and density  $n [\text{cm}^{-3}]$

Gases with a considerable acoustic noise background are encountered in various high temperature engineering systems, such as gas turbines, jet engines, rocket exhausts, etc. The theory presented permits calculation of the free energy  $\Delta F$  of the acoustic degrees of freedom in such systems, provided that the acoustic noise is in thermal equilibrium. Considerably larger free energy contributions are to be expected under nonequilibrium conditions, particularly if the acoustic fluctuations exhibit intensity levels corresponding to turbulence.

#### REFERENCES

1. L. D. Landau and E. M. Lifshitz, Statistical Physics (Pergamon Press, New York 1958).
2. V. P. Silin, J. Exper. Theor. Theor. Phys. USSR 23,649 (1952).
3. Y. H. Ichikawa, Progr. Theoret. Phys. 20, 715 (1958).
4. M. Abramotitz and I.A. Stegun, Handbook of Mathematical Functions (Dover Publ.,New York 1965).
5. P. J.W. Debye, "The Collected Papers of P.J.W. Debye" (Interscience, New York 1954).

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